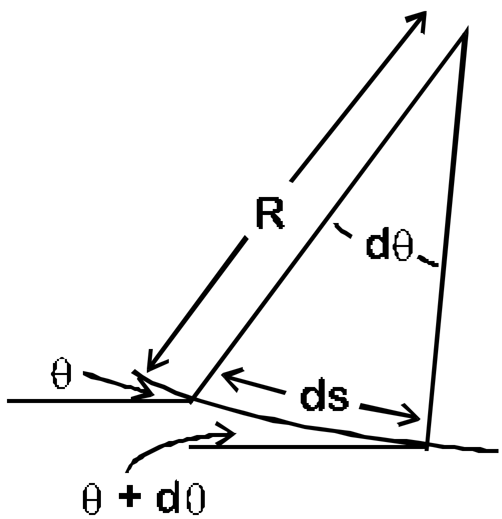
BENDING – SLOPE AND DEFLECTION

The bending equation revisited

Recall the bending equation

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}.$$

We can use the first equation, $\frac{M}{I} = \frac{E}{R}$ (1)



to find slopes and deflections. First, express R in relation to the slope θ. The diagram is of a section of beam bent into a circular arc. The angle between the radii, dθ, is equal to the change in the angle with the horizontal θ of a tangent to the beam. For an arc length s, we can use the relation 's = rθ' to suggest an expression

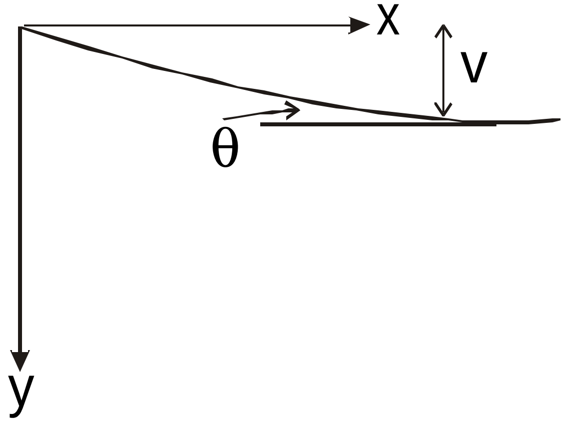
'ds = R dθ'

which relates to the diagram. However, note that θ is decreasing, and s increasing, so that the derivative $\frac{d\theta}{ds}$ is negative.

Therefore, when we re-arrange the above expression to get the derivative, we must take account of the sign and write

$$\frac{1}{R} = -\frac{d\theta}{ds} \qquad (2).$$

The deflection v and the slope θ are shown relative to the standard axis set in the diagram. The slope of the beam is equal to the gradient $\frac{dv}{dx}$. The gradient is also equal to the tangent of the angle



$\tan\theta = \frac{dv}{dx}$. For a small angle,
$\tan\theta \cong \theta$ and we may write

$$\theta = \frac{dv}{dx} \qquad (3).$$

The assumed small angle of deflection

also allows us to equate the arc length s with the co-ordinate x: s = x. We can therefore rewrite (2) as

$$\frac{1}{R} = -\frac{d\theta}{dx}$$

and now use (3) to conclude that

$$\frac{1}{R} = -\frac{d^2v}{dx^2} \qquad (4).$$

Now from equation (1),

$$\frac{1}{R} = \frac{M}{EI}$$

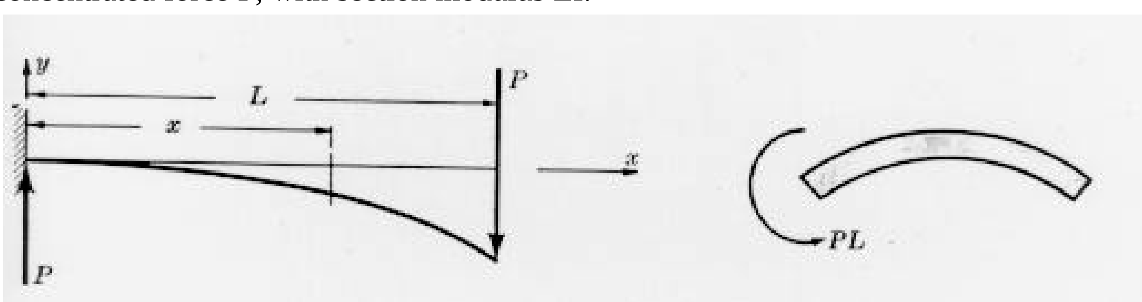
and we may rewrite (4) as

$$\frac{d^2v}{dx^2} = -\frac{M}{EI} \qquad (5).$$

Equation (5) is a differential equation in the deflection v, which can be solved to give both v and the slope $\frac{dv}{dx}$.

Example

Determine the deflection at every point of the cantilever beam subject to the single concentrated force P, with section modulus EI.



Solution

We need to use equation (5), so must first find the bending moment M. At the left-hand supported end, there is both a reaction P and a moment acting. It is therefore simpler to look at the moment produced by the force P at the right-hand end. For the point x, the moment M produced by the force P is given by

M = P(L-x).

We can now use (5):

$$\frac{d^2v}{dx^2} = -\frac{P(L-x)}{EI} = \frac{P(x-L)}{EI}.$$

Integrating the above gives

$$\frac{dv}{dx} = \frac{P}{EI}\left(\frac{x^2}{2} - Lx\right) + C_1 \qquad \text{(E1)}$$

where C_1 is a constant of integration. The equation above gives the slope of the beam. To obtain the deflection, integrate again:

$$v = \frac{P}{EI}\left(\frac{x^3}{6} - L\frac{x^2}{2}\right) + C_1x + C_2 \qquad \text{(E2)}$$

where C_2 is another constant of integration. To complete the solution, we need to find the unknown constants C_1 and C_2. These are obtained from the *boundary conditions*.

In this case, there are two boundary conditions needed to find the two constants. They both apply to the left-hand end of the beam. Firstly, the left-hand end is built-in, so the slope is constrained to be zero. This gives us a condition involving equation (E1):

$$0 = \frac{P}{EI}\left(\frac{x^2}{2} - Lx\right) + C_1.$$

At that left hand end, $x = 0$, and so the condition reduces to

$C_1 = 0$.

The other condition is that the deflection is zero at $x = 0$. This gives us, using equation (E2),

$$0 = \frac{P}{EI}\left(\frac{x^3}{6} - L\frac{x^2}{2}\right) + C_1x + C_2$$

which when $x = 0$ gives $C_2 = 0$. Putting these values back into (E2), the final solution is

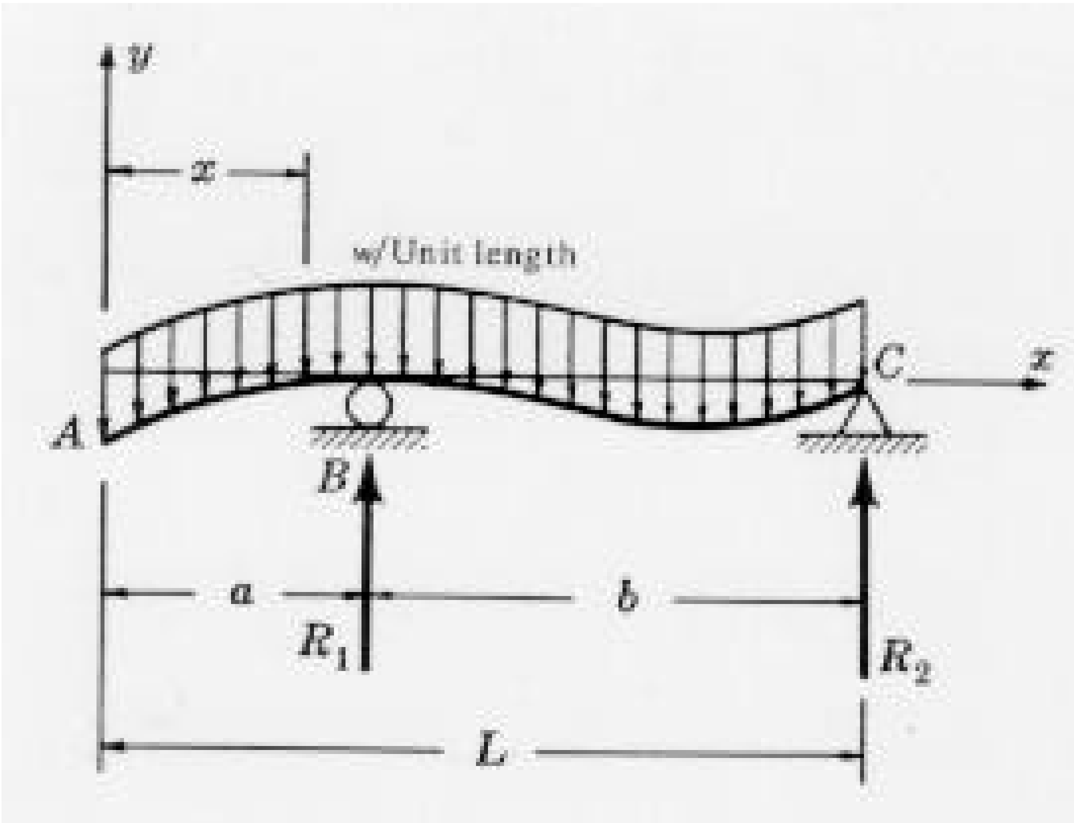
$$v = \frac{P}{EI}\left(\frac{x^3}{6} - L\frac{x^2}{2}\right)$$

i.e.

$$v = \frac{Px^2}{EI}\left(\frac{x}{6} - \frac{L}{2}\right).$$

Complex loading

The above method is called the ‘double integration method’. It is adequate for relatively simple problems, but is clumsy for anything other than simple loading. The following example illustrates the difficulty – finding the deflection at all points along this ‘overhanging beam’.



First we must find the bending moment along AB. For the point x, this is given by

$$M_{AB} = -\frac{wx^2}{2} \qquad \text{(M1).}$$

Using (5) it follows that, on AB,

$$EI\frac{d^2v}{dx^2} = \frac{wx^2}{2}.$$

Integrating as before gives

$$EI\frac{dv}{dx} = \frac{wx^3}{6} + C_1 \qquad (6)$$

and

$$EIv = \frac{wx^4}{24} + C_1x + C_2 \qquad (7).$$

For the other region of the beam BC, there is a different equation for the moment. Assuming a point x now within the region BC, the bending moment is now

$$M_{BC} = -\frac{wx^2}{2} + R_1(x-a) \qquad \text{(M2)}$$

where we have left the reaction as R_1 for now. This can now put into (5):

$$EI\frac{d^2v}{dx^2} = \frac{wx^2}{2} - R_1(x-a)$$

and integrated to give

$$EI\frac{dv}{dx} = \frac{wx^3}{6} - R_1\left(\frac{x^2}{2} - ax\right) + C_3 \qquad (8)$$

and again to give

$$EIv = \frac{wx^4}{24} - \left(\frac{x^3}{6} - \frac{ax^2}{2}\right) + C_3x + C_4 \qquad (9).$$

We have introduced two more constants of integration, making four in all. We therefore need four boundary conditions to evaluate them. These are

- In overhanging region AB, v = 0 when x = a.
- In region BC, v = 0 when x = a.
- On BC, when x = L, v = 0
- At B, slope $\frac{dv}{dx}$ for AB = slope $\frac{dv}{dx}$ for BC –i.e. no sudden kink.

Though possible, the procedure to find the solution for the two unknowns would be tedious, and it is clear that with yet more complex loading the method will become impractical. These difficulties can be avoided by using Macauley's method.

Step function / Macauley's method

We must first introduce the step function, denoted by the brackets '$\langle\ \rangle$'. For a variable x - a, we define

$$\langle x-a\rangle^n = \begin{cases} (x-a)^n & \text{for } x-a \geq 0 \\ 0 & \text{for } x - a < 0 \end{cases}$$

where n is any real number. This enables us to write down the bending moment for a beam such as the overhanging beam discussed above in a single equation. Equations (M1) and (M2) above can be put together into a single equation:

$$M = -\frac{wx^2}{2} + R_1\langle x-a\rangle^1 \qquad (10)$$

the inclusion of the exponent 1 being optional. The crucial property of the step function is that it can be integrated just like an ordinary pair of brackets:

$$\int_{x_1}^{x_2} \langle x-a \rangle^n dx = \left[\frac{\langle x-a \rangle^{n+1}}{n+1} \right]_{x_1}^{x_2} \qquad (11).$$

We now continue to solve the overhanging beam problem. Using equation (10) in (5), we have

$$EI\frac{d^2v}{dx^2} = \frac{wx^2}{2} - R_1 \langle x-a \rangle^1$$

which on integrating using (11) becomes

$$EI\frac{dv}{dx} = \frac{wx^3}{6} - R_1 \frac{\langle x-a \rangle^2}{2} + C_1 \qquad (12)$$

and again

$$EIv = \frac{wx^4}{24} - R_1 \frac{\langle x-a \rangle^3}{6} + C_1 x + C_2 \qquad (13).$$

Equations (12) and (13) apply for the whole beam and only involve two constants. We can find the constants by applying the boundary conditions:

- $v = 0$ at $x = a$.
- $v = 0$ at $x = L$

Using these values in equation (13) gives

$$0 = \frac{wa^4}{24} + C_1 a + C_2 \qquad (14)$$

$$0 = \frac{wL^4}{24} - R_1 \frac{\langle b \rangle^3}{6} + C_1 L + C_2 \qquad (15)$$

We now need to solve these for C_1 and C_2. Subtract (14) from (15) to eliminate C_2:

$$0 = \frac{w}{24}\left(L^4 - a^4\right) - R_1 \frac{b^3}{6} + C_1 (L-a)$$

where the step function has been evaluated for positive b. With $L - a = b$, dividing through by b gives

$$C_1 = R_1 \frac{b^2}{6} - \frac{w}{24b}(L^4 - a^4).$$

Now use (14) to get C_2:

$$C_2 = -\frac{wa^4}{24} - C_1 a\,.$$

Finally we find R_1 by taking moments about C to give

$$R_1 = \frac{wL^2}{2b}\,.$$

Putting in this value gives

$$\begin{aligned} C_1 &= \frac{wL^2 b}{12} - \frac{w}{24b}(L^4 - a^4) \\ C_2 &= -\frac{wa^4}{24} - \frac{wL^2 ab}{12} + \frac{wa}{24b}(L^4 - a^4) \end{aligned}\,.$$

To get the slope and deflection, these constants are inserted into equations (12) and (13).

Example. Use Macaulay's method to determine the equation of the deflection curve of the simply supported beam as shown.