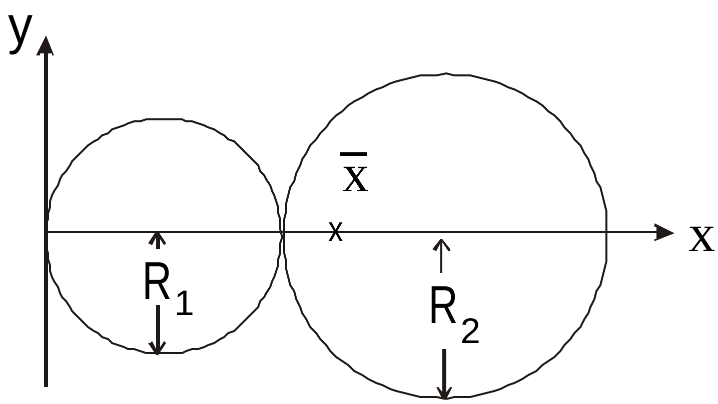
PROPERTIES OF AREAS

Centroid

The concept of the centroid is probably already familiar to you. For a plane shape with an obvious geometric centre, (rectangle, circle) the centroid is at the centre. If an area has an axis of symmetry, the centroid will be somewhere on that axis. For shapes that are made up of a number of elements with known centroids, we can find the position of the centroid of the whole such shapes by a process akin to taking moments. For example, consider a shape consisting of two circular areas touching. There is an axis of symmetry – the x axis – so the centroid will be somewhere on this axis. To find the position $\bar{x}$ of the centroid on this, take moments of areas about the y axis. The area of each circle is assumed to act at its centroid (its centre). The moment of the whole shape balances the moments from the individual circular areas:



$$\pi R_1^2 \times R_1 + \pi R_2^2 \times (2R_1 + R_2) = (\pi R_1^2 + \pi R_2^2) \times \bar{x} .$$

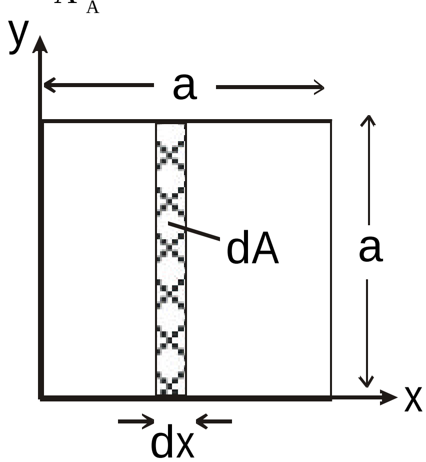
From this, we get

$$\bar{x} = \frac{R_1^3 + 2R_1R_2^2 + R_2^3}{R_1^2 + R_2^2} .$$

In general, and for an irregular shape, the definition of the centroid at position $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{1}{A}\int_A x dA$$
$$\bar{y} = \frac{1}{A}\int_A y dA \qquad (1)$$

where A is the area of the shape. The integrals mean 'integrate over the whole area A of the shape'. To see what this means, look at a simple example with an obvious answer – a square of side a. First find $\bar{x}$. What we are in effect doing is taking moments of area about the y axis. We therefore divide the area into strips that are at a constant distance from this axis, i.e, parallel to it. These strips can be thought of as having area dA. From the diagram, it is clear that dA = adx. We now replace dA in the first of equations (1) with adx. This gives us an integral which we can handle:

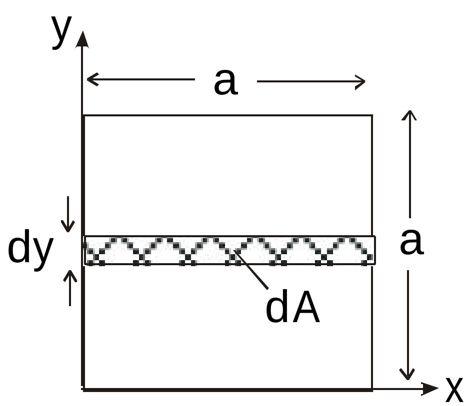


$$\bar{x} = \frac{1}{A}\int_0^a xadx$$

where the integral now has upper and lower limits defined by the range of x. Since A = a^2, this becomes

$$\bar{x} = \frac{a}{a^2}\int_0^a xdx = \frac{1}{a}\left[\frac{x^2}{2}\right]_0^a = \frac{a}{2}$$

which is the obvious result. We can do a similar exercise for the y co-ordinate. We are now taking moments about the x axis, so our choice of strip for the area dA is now parallel to the x axis. Now, dA = ady in the second of equations (1) The integral becomes

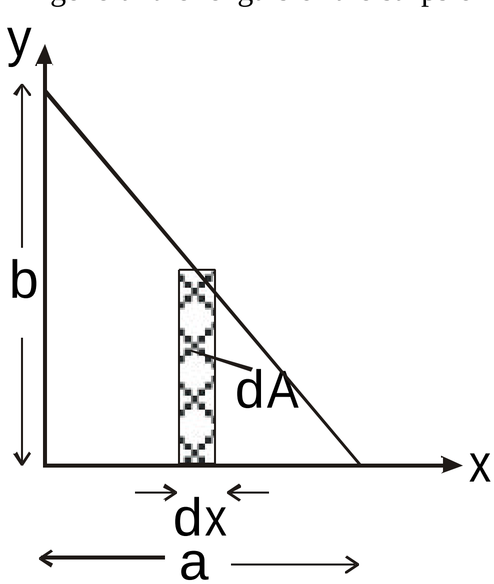


$$\bar{y} = \frac{a}{a^2}\int_0^a ydy = \frac{1}{a}\left[\frac{y^2}{2}\right]_0^a = \frac{a}{2}$$

again giving the expected result.

Centroid of a triangle

In general the lengths of the strips of material dA are not constant. Take the case of a right-angled triangle of height b and base a. Suppose we want to find $\bar{x}$. Then we need vertical strips as shown, with height varying with x. To determine the height, we need the equation for the triangle's oblique side. This is given by



$$y = b - \frac{b}{a}x .$$

The height h is just given by the y value, $h = b - \frac{b}{a}x$. We proceed as before to determine the area dA. Thus,

$$dA = hdx = \left(b - \frac{b}{a}x\right)dx .$$

Putting this in the first of the equations (1) gives

$$\bar{x} = \frac{1}{A}\int_0^a x\left(b - \frac{b}{a}x\right)dx = \frac{1}{\frac{1}{2}ab}\int_0^a bx - \frac{b}{a}x^2 dx = \frac{2}{ab}\left[\frac{bx^2}{2} - \frac{bx^3}{3a}\right]_0^a .$$

Putting in the limits gives

$$\bar{x} = \frac{2}{ab}\left[\frac{ba^2}{2} - \frac{ba^3}{3a}\right] = \frac{2b}{ab}\left[\frac{a^2}{2} - \frac{a^2}{3}\right] = 2a\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}a\,.$$

This is a well-known result.

Example

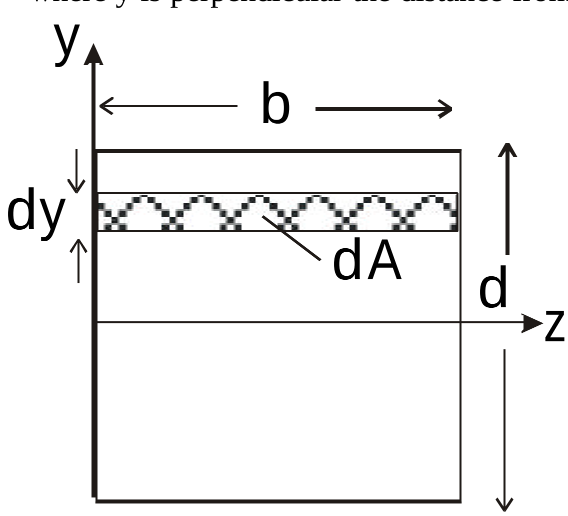
Show that $\bar{y} = \frac{1}{3}b$.

Second moment of area.

This is a quantity that will be required later for use in finding beam deflections and stresses. In standard bending axes, the plane of interest is labelled the y-z plane. Second moment of area is defined about a particular axis. There are standard formulae for common shapes (rectangle, circle) which are about the symmetry axes. Second moment of area I_z about an axis labelled the z-axis is defined as:

$$I_z = \int_A y^2 dA \qquad (2)$$

where y is perpendicular the distance from the z axis. Take the example of a rectangle, with the z axis an axis of symmetry (which must always pass through the centroid). We are taking moments about the z axis, so, as with the centroid, we choose strips parallel to this axis. Hence, dA = bdy and equation (2) becomes



$$I_z = \int_{-d/2}^{d/2} y^2 b dy$$

where the limits on the integral are defined by the range of y values over the shape. Integrating,

$$I_z = b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2} = b\left[\frac{d^3}{8.3} - \left(-\frac{d^3}{8.3}\right)\right] = \frac{bd^3}{12} \qquad (3).$$

This is the standard result for the rectangular section. Note that, for the I about a vertical y axis through the centroid, we can simply swap b for d and get

$$I_y = \frac{db^3}{12}\,.$$

These are the standard results for this section. Another useful standard result is for a solid circular section about a diameter, which is given by

$$I = \frac{\pi R^4}{4}$$

for a circle radius R.

Note that the units for second moment of area are length^4, i.e. mm^4 or m^4.

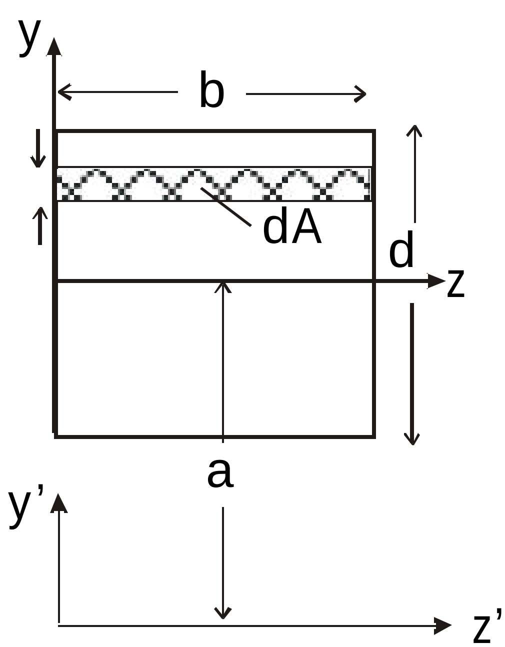
Parallel axis theorem

Sometimes we have the standard result for an area, which is for the second moment about an axis of symmetry, and we need to find the second moment about a different axis parallel to the original axis. The parallel axis theorem provides a convenient way of doing this. For an original z axis, and a parallel z' axis a perpendicular distance a from z, the original second moment I_z is related to the second moment $I_{z'}$ about z' by the expression:

$$I_{z'} = I_z + a^2 A \qquad (4).$$

Illustration

Find the second moment of area of the rectangular section about the z' axis shown.



Solution

Use equation (2), with y' the distance from the axis z'. The integral becomes

$$I_{z'} = \int_{a-d/2}^{a+d/2} by'^2 dy'.$$

On integration, we have

$$I_{z'} = b\left[\frac{y'^3}{3}\right]_{a-d/2}^{a+d/2} = \frac{b}{3}\left[(a+d/2)^3 - (a-d/2)^3\right]$$

which on multiplying out becomes

$$I_{z'} = \frac{b}{3}\left[a^3 + 3a^2d/2 + 3ad^2/4 + d^3/8 - (a^3 - 3a^2d/2 + 3ad^2/4 - d^3/8)\right]$$

$$= \frac{b}{3}\left[2d^3/8 + 3a^2d\right] = \frac{bd^3}{12} + a^2bd$$

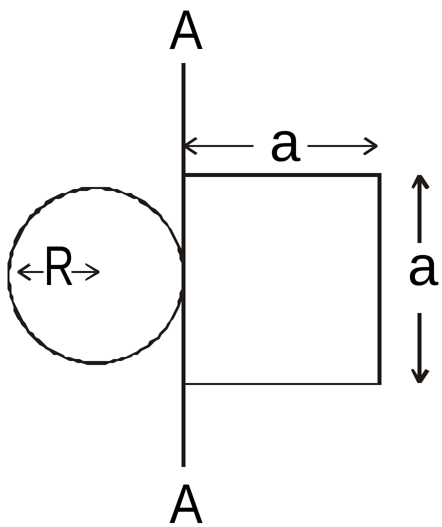
Note that we could get the same result using the standard result of equation (3) together with equation (4), given that A = bd.

Adding second moments.

Suppose we have a shape consisting of two separate areas. We can get the second moment for the whole shape by adding together the second moments for the two separate areas, *provided these two second moments are about the same axis*.

Example

Find the second moment of area for the shape illustrated about the axis AA.

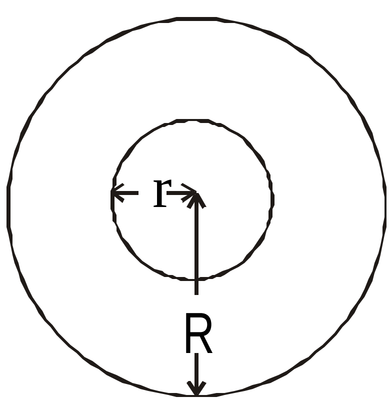


Outline solution

- Use standard formula to get I for square about its centre.
- Use Parallel Axis Theorem to get I for square about the axis AA – $I_{squareAA}$.
- Use standard formula to get I for circle about its diameter
- Use Parallel Axis Theorem to get I for circle about the axis AA – $I_{circleAA}$.
- The required result is $I_{squareAA} + I_{circleAA}$.

Hollow sections.

The additive nature of second moments makes possible a neat way of evaluating second moments of hollow sections. Suppose we have to find the second moment of the hollow circular section illustrated, about a diameter. Let the second moment of this section be I_{hollow}, and the second moments of solid circles of radius r and R respectively I_r and I_R. Since we are taking moments about the same axis, we may write



$I_{hollow} + I_r = I_R$.

Therefore,

$I_{hollow} = I_R - I_r$.

Using the standard result, it follows that for this section

$$I_{hollow} = \frac{\pi R^4}{4} - \frac{\pi r^4}{4} = \frac{\pi}{4}(R^4 - r^4).$$

The two problems given in the summary file for this chapter make use of this technique.