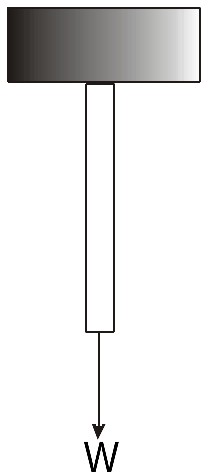
STATICALLY INDETERMINATE STRESS SYSTEMS

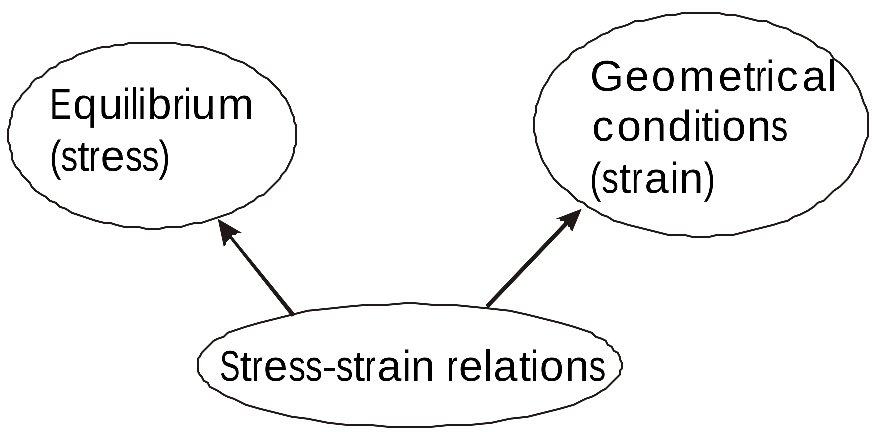
A statically *determinate* stress system

A simple example of this is a bar loaded by a weight, hanging in tension. The solution for the stress σ is simply $\sigma = W/A$ where A is the cross sectional area. Note that we did not need to know anything about the mechanical properties of the material of the bar to arrive at this solution. For this type of problem, where the forces acting upon the body are specified, the stress is independent of the mechanical behaviour. If we had defined the problem by specifying the displacement imposed on the end rather than the force, then we would need to know the material's modulus to evaluate the stress.



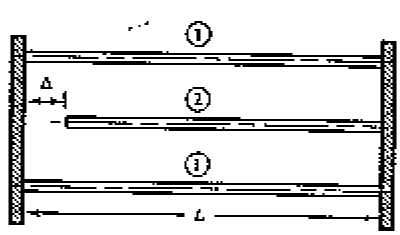
Structure of a Solid Mechanics problem

In general, we need to satisfy three conditions to obtain the solution to a Solid Mechanics problem. These are shown in the diagram. The above problem was a special case for which only the equilibrium condition was required. A geometrical condition may need to be satisfied also, and this will take the form of an equation involving strains. Finally, there must be a condition involving both stress and strain that relates the two – in our case, this is Hooke's Law.



Three-bar example.

This is a problem in which the geometrical condition is explicit. A system consists of two rigid end-plates, tied together by three horizontal bars as shown. Through a fabrication error, the central bar 2 is 0.0005L too short. All bars are of the same cross section and of steel with E = 210 GPa. Find the stress in each bar after the system has been mechanically pulled together so that the gap Δ is closed.



Ans. σ_1 = -35 MPa σ_2 = 70 MPa.

Solution

Let the length changes in the bars be δ_1 in the bars 1 and δ_2 in the bar 2. Note that δ_1 is a compression, and so negative, and δ_2 is a stretch and positive. The geometrical condition can be written as:

$-\delta_1 + \delta_2 = \Delta$ (1).

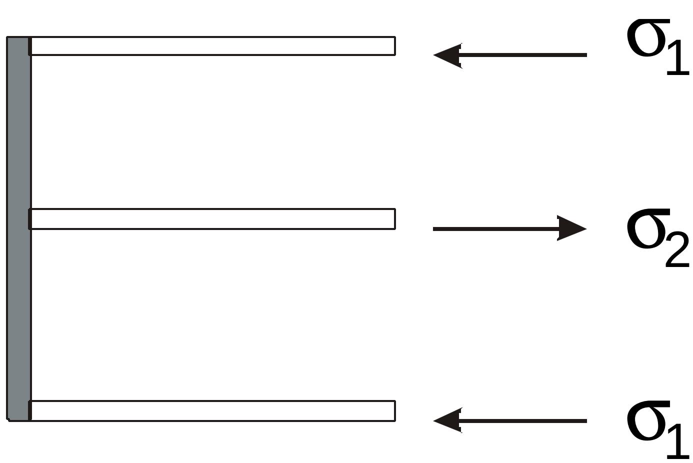
Now we use Hooke's Law for the bars 1 and the bar 2:

$$\varepsilon_1 = \frac{\delta_1}{L} = \frac{\sigma_1}{E} \Rightarrow \delta_1 = \frac{L\sigma_1}{E}$$
$$\varepsilon_2 = \frac{\delta_2}{L} = \frac{\sigma_2}{E} \Rightarrow \delta_2 = \frac{L\sigma_2}{E}$$

We now use the above two equations to substitute for δ_1 and δ_2 in (1):

$$\frac{L}{E}(\sigma_2 - \sigma_1) = \Delta \qquad (2).$$

So far we have the above equation containing two unknowns. We need another equation, which comes from the equilibrium condition. Make a vertical cut in the system to create a free body diagram:



Horizontal equilibrium gives

$A\sigma_1 + A\sigma_2 + A\sigma_2$ c= 0

Where A is the cross section of each bar. Hence,

$\sigma_2 = -2\sigma_1$ (3).

Equations (2) and (3) form a system of two equations in two unknowns. To solve, substitute for σ_2 from (3) into (2):

$$\frac{L}{E}(-3\sigma_1) = \Delta .$$

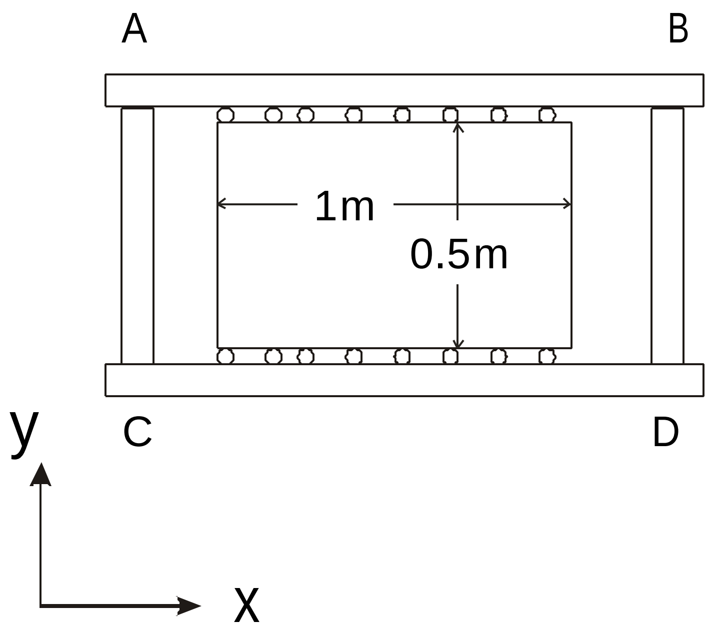
From this we obtain

$$\sigma_1 = -\frac{\Delta E}{3L} = -\frac{0.0005 \times 210 \times 10^3}{3} \text{ Mpa}$$

which is –35 Mpa.

Equation (3) now gives σ_2 =70 Mpa.

Two-dimensional example



A 10 mm thick sheet is held within a rigid frame, which prevents any movement in the y direction. It is strained by 1% along the x direction. It has a Young's modulus of 1 GPa .and a Poisson's ratio of 0.4. What is the sum of the forces acting in the members AC and BD? Ans. 47.62kN

Thermal strains

Recall Hooke's Law (week 4) for the case of varying temperature:

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right) + \alpha(T - T_0)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_z + \sigma_x)\right) + \alpha(T - T_0)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right) + \alpha(T - T_0)$$

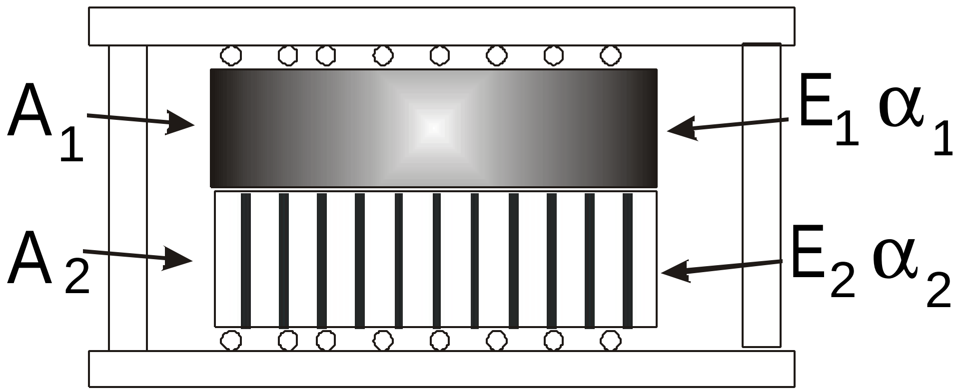
When there is no stress along y or z, it reduces to

$$\varepsilon_x = \frac{1}{E}\sigma_x + \alpha(T - T_0) \qquad (4)$$

where α is the coefficient of expansion and $T - T_0$ the temperature rise.

Bimetallic strip example.

A bimetallic strip is heated up and not allowed to bend. What stresses are set up as a result?



Solution

Let the cross-sectional areas, moduli and coefficients of expansion be as indicated. Suppose the bimetallic strip is subject to a temperature rise T. Then the stresses and strains in each strip are related by Hooke's Law (4) above:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E}\sigma_1 + \alpha_1 T \\ \varepsilon_2 &= \frac{1}{E}\sigma_2 + \alpha_2 T \end{aligned} \qquad (5).$$

We now apply the geometrical condition – the strips are bonded together and there is no bending. This means that there is a constant strain in each strip, and the strains in the two strips are equal:

$$\varepsilon_1 = \varepsilon_2 \qquad (6).$$

The third condition (see page 1!) is equilibrium. The total horizontal force on the bimetallic strip is zero:

$$\sigma_1 A_1 + \sigma_2 A_2 = 0 \qquad (7).$$

Equations (5), (6) and (7) are four in number, so should be soluble for the unknowns σ_1,ε_1, σ_2 and ε_2. We are only interested in the stresses so proceed by eliminating the strains. Using (6) in (5) gives

$$\frac{1}{E_1}\sigma_1 + \alpha_1 T - \frac{1}{E_2}\sigma_2 - \alpha_2 T = 0 \qquad (8).$$

Now from (7), $\sigma_2 = -\frac{\sigma_1 A_1}{A_2}$. Substituting in (8) gives

$$\frac{1}{E_1}\sigma_1 + \alpha_1 T - \frac{1}{E_2}\left(-\frac{\sigma_1 A_1}{A_2}\right) - \alpha_2 T = 0 .$$

This now needs to be rearranged to give σ_1. Put the two σ_1 terms together:

$$\sigma_1\left(\frac{1}{E_1} + \frac{A_1}{E_2 A_2}\right) + (\alpha_1 - \alpha_2)T = 0$$

To give the stress

$$\sigma_1 = -\frac{(\alpha_1 - \alpha_2)T}{\frac{1}{E_1} + \frac{A_1}{A_2 E_2}}$$

which looks a bit nicer if we multiply top and bottom by $E_1 E_2 A_2$:

$$\sigma_1 = -\frac{E_1 E_2 A_2 (\alpha_1 - \alpha_2)T}{E_2 A_2 + E_1 A_1} .$$

Notice that when the expansion coefficients are equal, there is no stress – as you would expect. Finally, we get σ_2 using the expression obtained above, $\sigma_2 = -\frac{\sigma_1 A_1}{A_2}$. This gives us

$$\sigma_2 = \frac{E_1 E_2 A_1 (\alpha_1 - \alpha_2)T}{E_1 A_1 + E_2 A_2} .$$