

## STRESS-STRAIN RELATIONS

### Strain

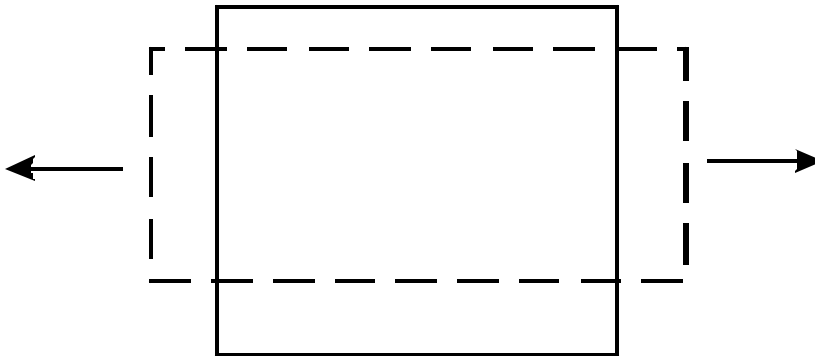
Strain is related to change in dimensions and shape of a material. The most elementary definition of strain is when the deformation is along one axis:

$$\text{strain} = \frac{\text{change in length}}{\text{original length}} \quad (1).$$

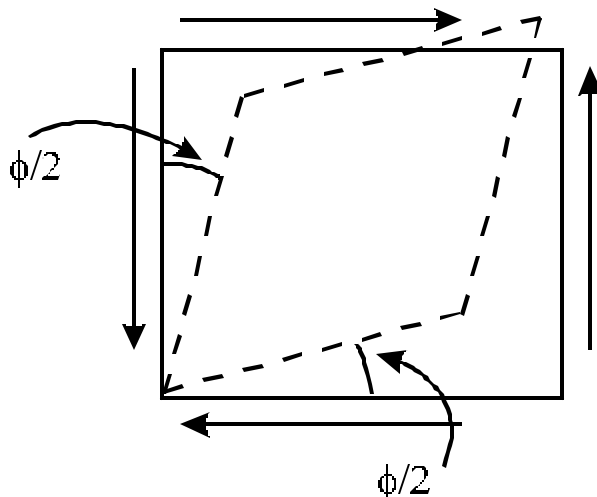
When a material is stretched, the change in length and the strain are positive. When it is compressed, the change in length and strain are negative. This conforms with the signs of the stresses which would accompany these strains, tensile stresses being positive and compressive stresses negative. This definition refers to what are termed *normal strains*, which change the dimensions of a material but not its shape; in other words, angles do not change. In general, there are normal strains along three mutually perpendicular axes.

By contrast, strains which involve no length changes but which do change angles are known as *shear strains*. The measure of strain is the change in the angle. Normal and shear strains are illustrated below.

### NORMAL



### SHEAR



Strained shapes are the broken lines. The shear strain is the total change in the angle  $\phi$ .

For normal strains the usual symbol is  $\epsilon$ . The same system of subscripts is used as for stresses. With respect to  $(x,y,z)$  axes we have strains  $\epsilon_{xx}$ ,  $\epsilon_{yy}$  and  $\epsilon_{zz}$ , with, as with stresses, the first subscript giving the normal to the plane which is being displaced, and the second the direction of the displacement. Since the subscripts are always repeated, we sometimes just use a single subscript. Conventionally, the symbol for shear stress is  $\gamma$  and the same subscripting system gives rise to strains such as  $\gamma_{xy}$ ,  $\gamma_{yz}$ , and  $\gamma_{zx}$ .

Other quantities that are associated with strain are *displacements*. These are simply the distance moved by any point of material. Usually, the displacements in the  $(x,y,z)$  directions are denoted respectively by  $(u,v,w)$ . When a strain is present, the displacements must vary from point to point. It is easy to see that normal strains are given by

$$\begin{aligned}\epsilon_{xx} &= \frac{\partial u}{\partial x} \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} \\ \epsilon_{zz} &= \frac{\partial w}{\partial z}\end{aligned}\quad (2).$$

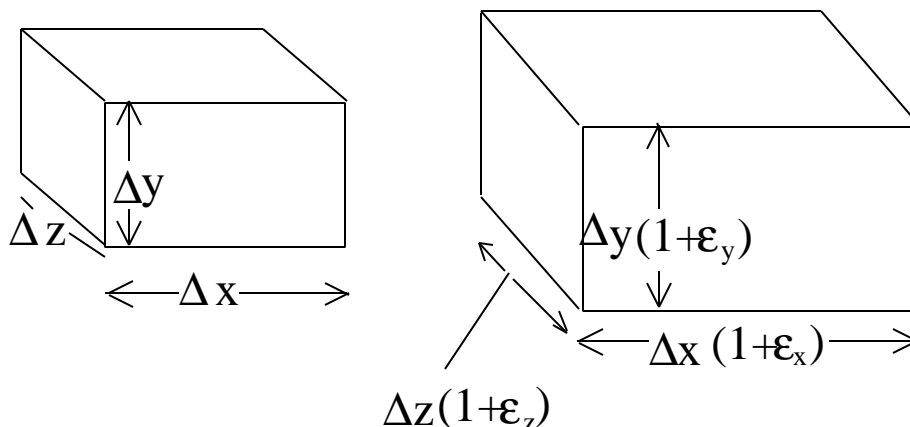
This is a more rigorous definition of normal strain than (1) above.

In this module, strains are always *small*. This enables us to use simple theory. In this context, small means that the square of the strain is negligible in comparison to the strain itself, e.g.  $\epsilon^2 \ll \epsilon$ .

In general, the action of stresses will cause material to change in volume. Only the normal stresses are associated with volume strain. We define volumetric strain  $e$  as:

$$e = \frac{\text{change in volume}}{\text{original volume}} \quad (3).$$

The volumetric strain is simply related to the normal strains. Consider the rectangular solid illustrated. The original shape is on the left and the deformed shape on the right. The original volume is given by  $V_0 = \Delta x \Delta y \Delta z$ . For the deformed body on the right,



the volume is given by

$$V = \Delta x(1 + \epsilon_x)\Delta y(1 + \epsilon_y)\Delta z(1 + \epsilon_z) = \Delta x\Delta y\Delta z(1 + \epsilon_x + \epsilon_y + \epsilon_z + \epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x + \epsilon_x\epsilon_y\epsilon_z)$$

We now recall that the strains are small, so that we can neglect all the high order terms so that

$$V \approx \Delta x\Delta y\Delta z(1 + \epsilon_x + \epsilon_y + \epsilon_z) = V_0(1 + \epsilon_x + \epsilon_y + \epsilon_z)$$

Now, we can use the definition of volumetric strain (3) which becomes

$$e = \frac{V - V_0}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z \quad (4).$$

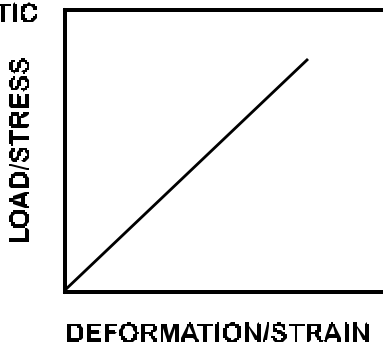
This is the simple relationship we seek, which applies at small strains only.

### Stress-strain relations

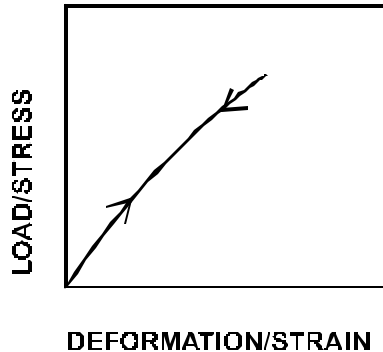
Materials undergo strain when they are subject to stress. The relationship between stress and strain is different for different materials, and can be appreciated by plotting stress against strain. Suppose a material specimen is subject to a uniaxial tensile load.

It will deform in a manner characteristic of the material. Examples of possible behaviour are shown here. In this module, we assume that materials obey the first type of behaviour, *linear elastic*.

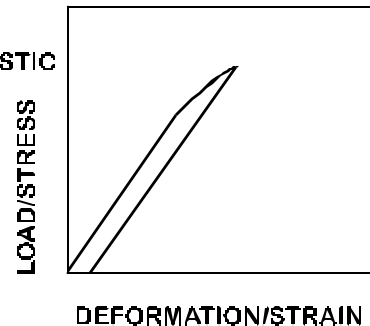
#### LINEAR ELASTIC



#### NONLINEAR ELASTIC



#### ELASTIC-PLASTIC



This means that stresses and strains are assumed to be related by *Hooke's Law*. In its simplest form, Hooke's Law can be stated as:

$$\frac{\text{stress}}{\text{strain}} = \text{constant} \quad (5)$$

where the constant is known as an *elastic modulus* or simply a *modulus*.

#### *Normal strains and stresses*

For normal stress along the x direction only, (5) becomes

$$\frac{\sigma_x}{\epsilon_x} = E$$

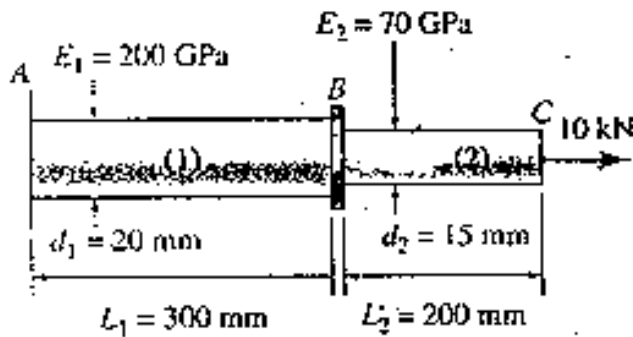
or

$$\epsilon_x = \frac{1}{E} \sigma_x \quad (6)$$

where the constant E is known as *Young's modulus*.

#### Example

Determine the total elongation of the member ABC if AB is steel ( $E = 200 \text{ GPa}$ ) and BC is aluminium ( $E = 70 \text{ GPa}$ ).



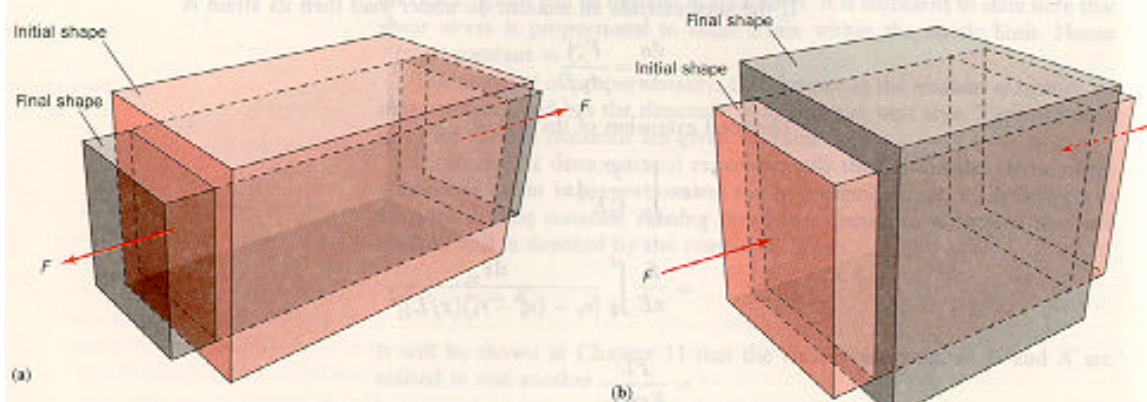
#### Example

A solid of revolution is of radius R at one end and tapers so that its radius r at an axial distance x from the end is

$$r(x) = R e^{-\alpha x}$$

It is subject to an axial force F. If its initial length is L, and its modulus E, what is its stretched length?

Suppose the stress is tensile, and the specimen of material is stretched along x. Then, it will get thinner across the direction of stretching, in the y and z directions, as in (a).



The strains will be negative along y and z. They are related to the strain along x by a positive constant  $\nu$ :

$$\epsilon_y = \epsilon_z = -\nu \epsilon_x .$$

$\nu$  is known as Poisson's ratio. Using (6) we can see that the stress along x produces transverse strains:

$$\epsilon_y = \epsilon_z = -\nu \frac{\sigma_x}{E} .$$

This is true when we only have stress along x. In general, we have stress along y and z also. These other stresses will cause transverse strains also. The transverse strains along the same axis arising from different stresses simply add together. When three stresses act, the general result is

$$\begin{aligned} \epsilon_x &= \frac{1}{E} \left( \sigma_x - \nu(\sigma_y + \sigma_z) \right) \\ \epsilon_y &= \frac{1}{E} \left( \sigma_y - \nu(\sigma_z + \sigma_x) \right) \\ \epsilon_z &= \frac{1}{E} \left( \sigma_z - \nu(\sigma_x + \sigma_y) \right) \end{aligned} \tag{7} .$$

This is three-dimensional Hooke's Law for normal stresses.

### *Shear strains and stresses*

These relationships are less complex than those for normal strains. Equation (5) still applies, and each strain component is proportional to the corresponding stress component. We can write three equations similar to (6):

$$\begin{aligned}\gamma_{xy} &= \frac{1}{G} \tau_{xy} \\ \gamma_{yz} &= \frac{1}{G} \tau_{yz} \\ \gamma_{zx} &= \frac{1}{G} \tau_{zx}\end{aligned}\tag{8}.$$

The constant  $G$  is the shear modulus. Equations (7) and (8) constitute the full statement of Hooke's Law in three dimensions.

### *Volumetric strain and hydrostatic stress.*

These two quantities are linearly related. The version of equation (5) is

$$e = \frac{1}{K} \sigma \tag{9}$$

where  $e$  is the volumetric strain,  $\sigma$  the hydrostatic stress and the constant  $K$  the *bulk modulus*.

### *Summary*

The quantities  $E$ ,  $\nu$ ,  $G$  and  $K$  introduced above are elastic constants. They are not all independent. The first two are sufficient to fully define any linearly elastic material.

### *Plane strain and plane stress*

Equations (7) and (8) give the fully three-dimensional form of Hooke's Law. Many problems can be approximated by a two-dimensional approach, by assuming that stress or strain varies only in a single plane, usually the  $x$ - $y$  plane. There are two ways to create such an approximation: by assuming that the force normal to the plane is zero – the plane stress assumption; or by assuming that the strain normal to the plane is zero – the plane strain assumption. The two approximations We shall outline the plane stress type of analysis first.

We simply take equations (7) and put  $\sigma_z = 0$ . Then we have

$$\begin{aligned}\epsilon_x &= \frac{1}{E} (\sigma_x - \nu \sigma_y) \\ \epsilon_y &= \frac{1}{E} (\sigma_y - \nu \sigma_x) \\ \epsilon_z &= -\frac{\nu}{E} (\sigma_x + \sigma_y)\end{aligned}\tag{10}.$$

Finally, the zero force in the z direction implies that  $\gamma_{yz} = \gamma_{zx} = 0$ . Therefore only the first of equations (8) is nontrivial:

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} \quad (11).$$

Equations (10) and (11) define the plane stress formulation.

The plane strain analysis is a little more complex. Putting  $\epsilon_z = 0$  in the last equation (7) gives an expression for  $\sigma_z$ :

$$\sigma_z = \nu(\sigma_x + \sigma_y).$$

This can be substituted into the first two equations (7) to give expressions involving  $\sigma_x$  and  $\sigma_y$  only. The first equation becomes

$$\begin{aligned} \epsilon_x &= \frac{1}{E} (\sigma_x - \nu(\sigma_y + \nu(\sigma_x + \sigma_y))) \\ &= \frac{1}{E} (\sigma_x (1 - \nu^2) - \nu(\sigma_y (1 + \nu))) \\ &= \frac{1 - \nu^2}{E} \left( \sigma_x - \frac{\nu}{1 - \nu} \sigma_y \right) \end{aligned} \quad (12).$$

We can define a plane strain modulus  $E'$ , such that  $E' = \frac{E}{1 - \nu^2}$ , and a quantity  $\nu'$  such that  $\nu' = \frac{\nu}{1 - \nu}$ . Then we can write the above equation as

$$\epsilon_x = \frac{1}{E'} (\sigma_x - \nu' \sigma_y) \quad (13)$$

which resembles the first of equations (10). Similarly, for the y direction

$$\epsilon_y = \frac{1}{E'} (\sigma_y - \nu' \sigma_x) \quad (14).$$

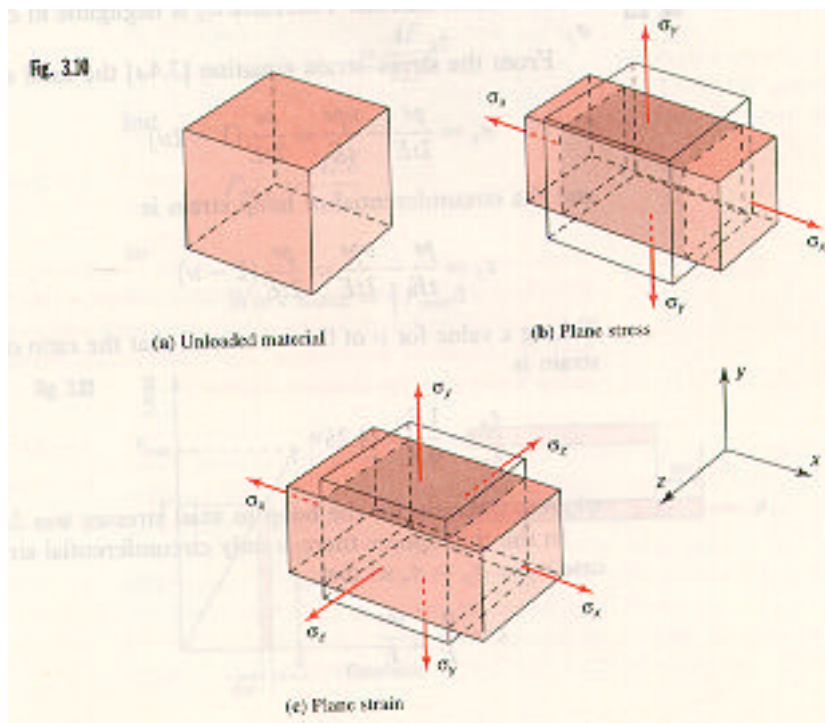
We complete the formulation with what we already know:

$$\epsilon_z = 0 \quad (15)$$

and note that equation (11) still applies in plane strain.

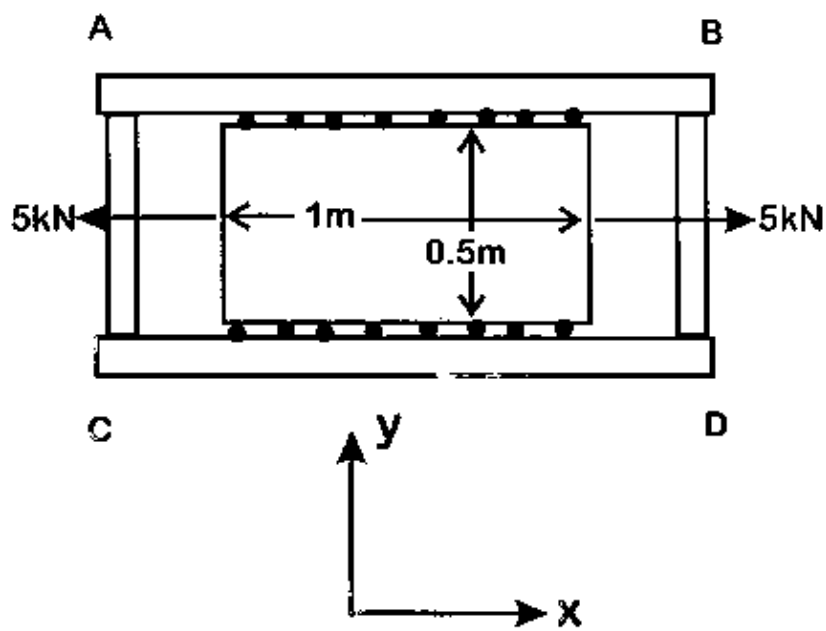
Equations (11) and (13) – (15) complete the plane strain relations.

The states of plane stress and plane strain, for material stretched along x, are illustrated.



### Example

The illustrated device stretches a thin sheet of material while holding the width constant. The sheet is 10 mm thick, and is linearly elastic with Poisson's ratio  $\nu = 0.3$ . If the pulling force along x is 5 kN, what are the forces in the members AC and BD?





### Thermal strains

A rise in temperature causes materials to expand. In simple cases, the increase in length is simply proportional to the temperature rise, via a constant of proportionality  $\alpha$ , the coefficient of expansion. The temperature rise multiplied by  $\alpha$  gives the increase in strain caused by the temperature – the thermal strain. For an initial temperature  $T_0$  and an operating temperature  $T$ , the thermal strain is  $\alpha(T - T_0)$ . These strains are simply added to the strains caused by the stresses. For varying temperature, equations (7) are revised as

$$\begin{aligned}\epsilon_x &= \frac{1}{E} (\sigma_x - \nu(\sigma_y + \sigma_z)) + \alpha(T - T_0) \\ \epsilon_y &= \frac{1}{E} (\sigma_y - \nu(\sigma_z + \sigma_x)) + \alpha(T - T_0) \\ \epsilon_z &= \frac{1}{E} (\sigma_z - \nu(\sigma_x + \sigma_y)) + \alpha(T - T_0)\end{aligned}\tag{9}.$$

### Example

Two bars are joined together and attached at supports as shown. The left bar is brass for which  $E = 90 \text{ Gpa}$ ,  $\alpha = 20 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$  and the right bar is aluminium for which.  $E = 70 \text{ Gpa}$ ,  $\alpha = 25 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$ . The cross sectional area of the brass bar is  $500 \text{ mm}^2$  and that of the aluminium bar is  $750 \text{ mm}^2$ . The system is initially stress free and the temperature then drops by  $20^\circ\text{C}$ . The bars have negligible weight.

- If the supports are unyielding, find the normal stress in each bar.
- If the right supports yields  $0.1\text{mm}$ , find the normal stress in each bar.

Ans. (a) brass  $41 \text{ Mpa}$ , aluminium  $27.33 \text{ Mpa}$ ; (b) brass  $28.4 \text{ Mpa}$ , aluminium  $19 \text{ Mpa}$ .

