

TRANSFORMATION OF STRESSES

Nature of the problem

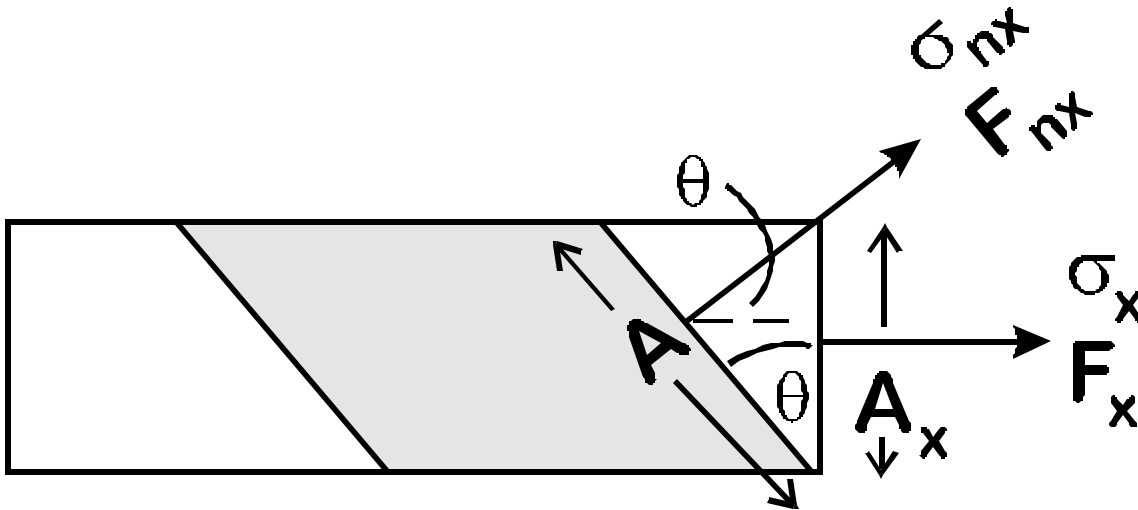
We have developed methods of dealing with stresses which are based on (x,y,z) axes. Stresses are defined relative to a particular axis set. The numbers that represent the stress depend on the directions chosen for the axis set. It is often necessary to change from one axis set to another, which is equivalent to resolving the stress along a direction at an oblique angle to the original. Stresses are directly related to forces, and so you might expect that the procedures for resolving stresses would be the same as those for forces. Things are not this simple, since when we must take into account the change in the area on which the force acts as we rotate the axis set.

Analysis

We confine the analysis to two dimensions. Suppose we have a stress system with respect to x and y axes with normal and shear components σ_x , σ_y and τ_{xy} . We wish to find the stresses acting on a surface tilted through an angle θ with respect to the original. First we find the stress normal to the tilted surface. We consider in turn the effects of the three stress components σ_x , σ_y and τ_{xy} .

For stress σ_x

We envisage that the stress σ_x is caused by a force F_x acting normal to a plane of area A_x .



First transform force F_x normal to the oblique plane:

$$F_{nx} = F_x \cos \theta$$

Now, express this in terms of stress (force = stress \times area):

$$F_{nx} = \sigma_x A_x \cos \theta$$

Now translate this into a stress (Stress = force/area):

$$\sigma_{nx} = \frac{\sigma_x A_x}{A} \cos \theta$$

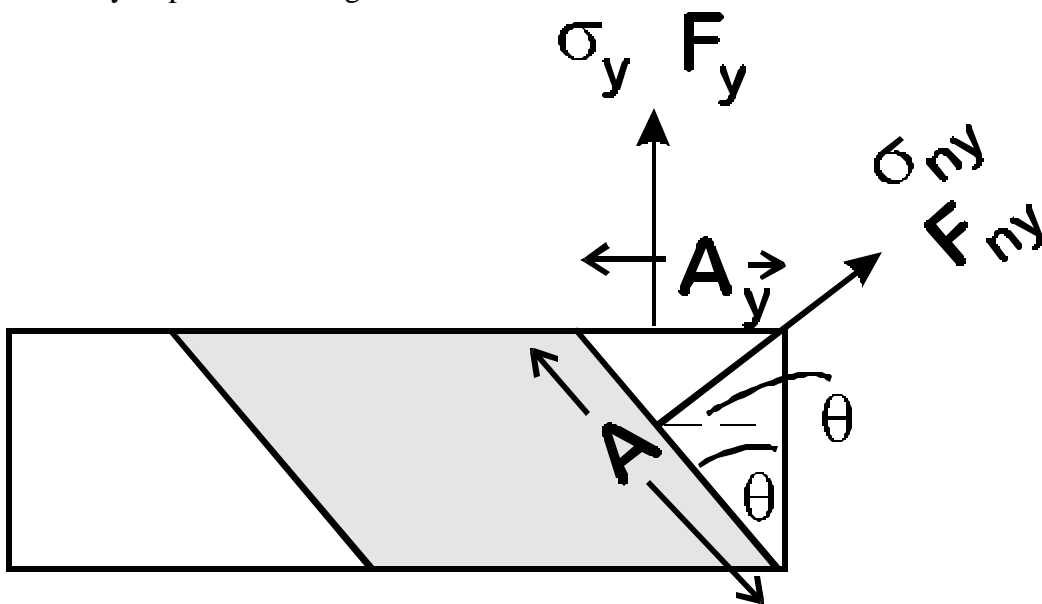
Relationship between areas: $A_x = A \cos \theta$

So stress transformation is

$$\sigma_{nx} = \sigma_x \cos^2 \theta \quad (1).$$

For stress σ_y

The analysis proceeds along similar lines.



Transform force F_y $F_{ny} = F_y \sin \theta$

Force = stress \times area: $F_{ny} = \sigma_y A_y \sin \theta$

Stress = force/area: $\sigma_{ny} = \frac{\sigma_y A_y}{A} \sin \theta$

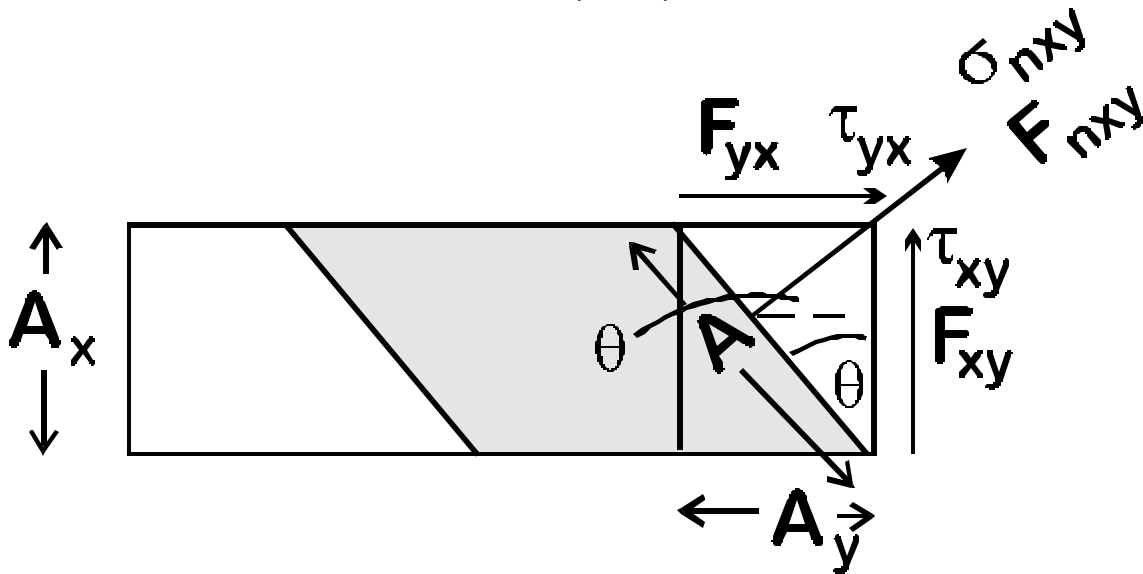
Relationship between areas: $A_y = A \sin \theta$

So stress transformation is

$$\sigma_{ny} = \sigma_y \sin^2 \theta \quad (2).$$

For stress τ_{xy}

We must take account of both components τ_{xy} and τ_{yx} .



Transform forces F_{yx}, F_{xy} $F_{nxy} = F_{yx} \cos \theta + F_{xy} \sin \theta$

Force = stress \times area: $F_{nxy} = \tau_{yx} A_y \cos \theta + \tau_{xy} A_x \sin \theta$

Stress = force/area: $\sigma_{nxy} = \frac{1}{A} (\tau_{yx} A_y \cos \theta + \tau_{xy} A_x \sin \theta)$

Relationship between areas: $A_x = A \cos \theta; A_y = A \sin \theta$

So stress transformation is

$$\sigma_{nxy} = 2\tau_{xy} \sin \theta \cos \theta \quad (3)$$

(since $\tau_{xy} = \tau_{yx}$).

Total normal stress

The total normal stress on oblique plane as a result of $\sigma_x, \sigma_y, \tau_{xy}$ acting together is, adding equations (1), (2) and (3):

$$\sigma_n = \sigma_{nx} + \sigma_{ny} + \sigma_{nxy}$$

which becomes

$$\sigma_n = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta \quad (4)$$

This is a useful transformation equation. It is often re-expressed using the double angle formulae

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

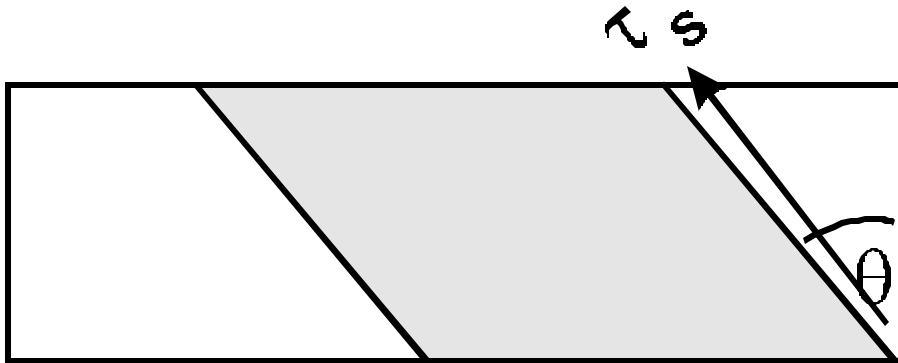
$$2 \sin \theta \cos \theta = \sin 2\theta$$

to give

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \quad (5).$$

Shear stress on the oblique plane

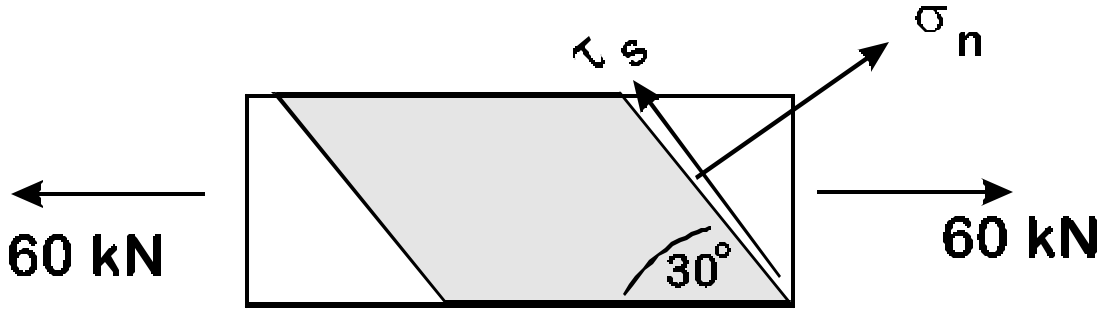
Rather than go through the analysis, we just show the result for the shear stress τ_s acting on the oblique plane.



$$\tau_s = -\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta \quad (6).$$

Example

A bar of cross section 850 mm^2 is acted upon by tensile forces of 60 kN applied to each end of the bar. Determine the normal and shear stresses acting on a plane inclined at 30° to the direction of loading.



Mohr's circle

This is a diagrammatic tool for transforming stresses. It takes the form of a plot of shear stress against normal stress. Equations (5) and (6) provide expressions for the co-ordinates σ_n and τ_s . Rearranging (5) gives

$$\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \quad (7)$$

and rewriting (6) gives

$$\tau_s = -\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta \quad (6)$$

We can demonstrate that the relationship between the normal and shear stress is described by a circle in the $\sigma - \tau$ plane by squaring and adding the above pair of equations.

$$(7)^2 + (6)^2:$$

$$\left(\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y) \right)^2 + \tau_s^2 = \frac{1}{4}(\sigma_x - \sigma_y)^2 \cos^2 2\theta + (\sigma_x - \sigma_y) \tau_{xy} \sin 2\theta \cos 2\theta + \tau_{xy}^2 \sin^2 2\theta + \frac{1}{4}(\sigma_x - \sigma_y)^2 \sin^2 2\theta - (\sigma_x - \sigma_y) \tau_{xy} \sin 2\theta \cos 2\theta + \tau_{xy}^2 \cos^2 2\theta$$

Since $\sin^2 2\theta + \cos^2 2\theta = 1$, the right hand side becomes independent of θ . Replacing σ_n and τ_s by just σ and τ , we get

$$\left(\sigma - \frac{1}{2}(\sigma_x + \sigma_y) \right)^2 + \tau^2 = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 \quad (8).$$

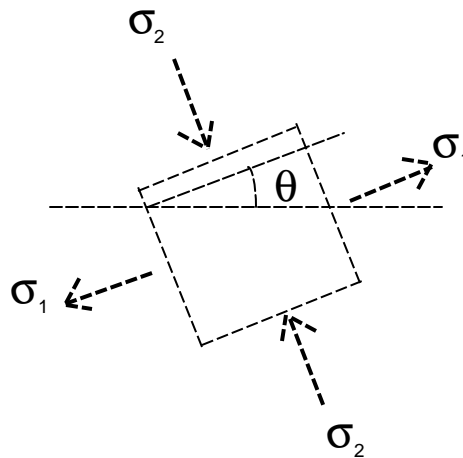
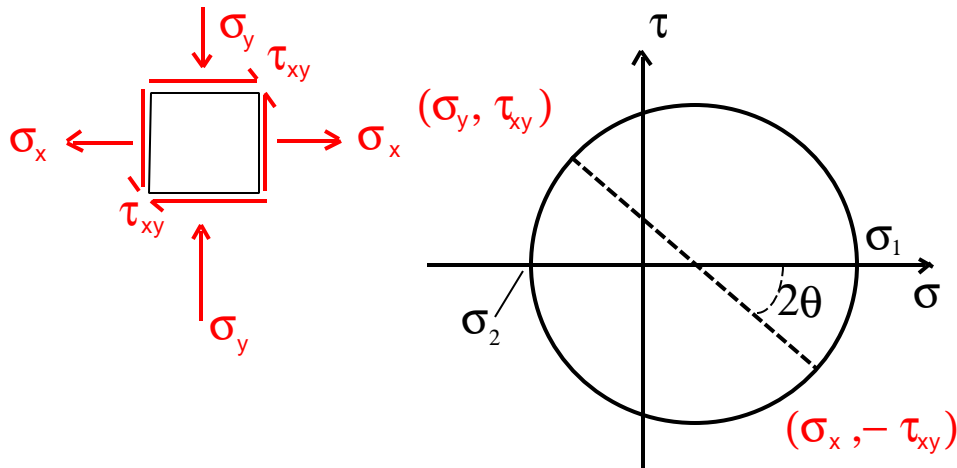
This is the equation of a circle in the $\sigma - \tau$ plane with centre at

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y), \quad \tau = 0 \text{ and radius } \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}.$$

The co-ordinates for any angle θ are given by equations (6) and (7). Because of the double angles in these equations, a rotation of angle θ in real space corresponds to an angle 2θ around the circle. A standard procedure can be adopted to construct the circle, which can be used to read off the stresses at any angle.

Construction of circle

In the example below there are normal and shear stresses acting on the square element shown. We construct the circle by first plotting the two ends of its diameter. Since we get from one face of the square to the adjacent face by turning through an angle of $\pi/2$, it follows that, on the circle, the points representing these two faces are separated by $2 \times \pi/2 = \pi$, i.e. they are opposite ends of a diameter. To plot the first point, use the top face of the element and plot the point (σ_y, τ_{xy}) . From the direction of the arrow, it is indicated that σ_y is negative. For the shear stress, the rule is that a component that



would cause a clockwise rotation is positive, anticlockwise negative. We now travel clockwise one right angle around the element to get the point $(\sigma_x, -\tau_{xy})$. This is plotted on the circle diagram as the other end of the diameter. The circle is now

defined. To find the stress on a face at an angle θ to the vertical right-hand face of the element, we start on the diagram from the point representing that face and travel around the circle through an angle 2θ in the same direction as the rotation in the 'real space' element.

Notice that the circle always cuts the τ axis at two points. This means that, for any stress field, we can always find two directions for which the shear stress is zero, and that these directions are an angle $\pi/2$. They are known as *principal directions*, and the corresponding stresses (σ_1 and σ_2) are the *principal stresses*. Since we know the radius of the Mohr circle and the position of its centre, both given above, it is easy to see that

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2} \\ \sigma_2 &= \frac{1}{2}(\sigma_x + \sigma_y) - \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}\end{aligned}\quad (9).$$

The angle θ between the principal directions and the x axis is given by

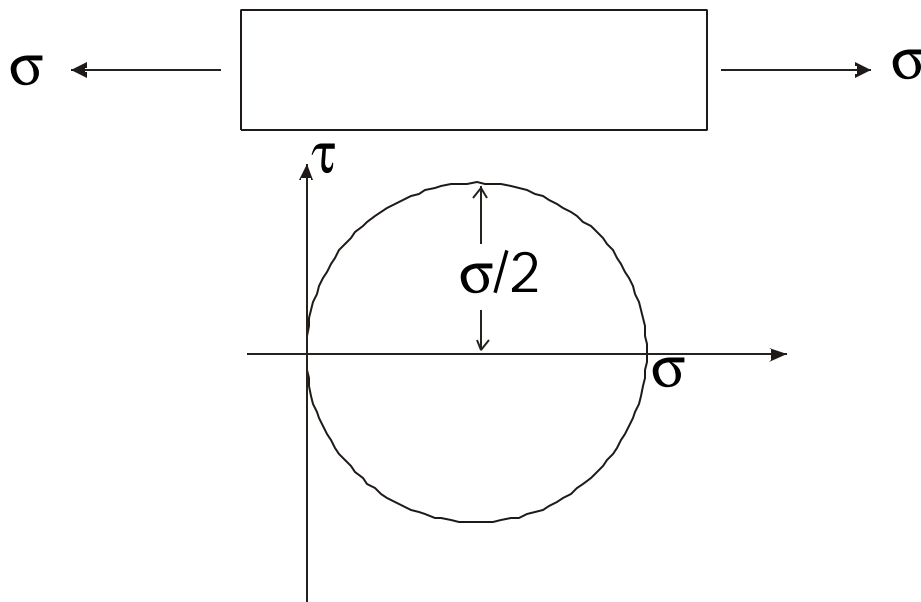
$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

which gives two values of θ that differ by $\pi/2$, corresponding to the two perpendicular principal directions.

Example

Construct the Mohr circle for uniaxial tension.

Solution



Example

Construct Mohr circle for pure shear. What are the principal stresses and principal directions?

Example

Find, for the stress set-up in the diagram, the principal stresses and directions, and also the maximum shear stress.

