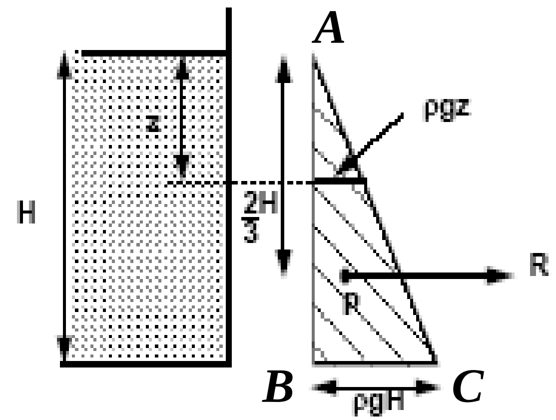
CHAPTER 3

Static Forces on Surfaces
Buoyancy

Dr Yunes Mogheir



OBJECTIVES

1. Compute the hydrostatic pressures and forces on submerged surfaces in a static fluid.
2. Use the principle of static equilibrium to solve for the forces involves in buoyancy problems.
3. Define the condition for stability of submerged and floating bodies.

INTRODUCTION

- When a surface is submerged in a **Fluid,** forces develop on the surface due to the fluid**.**

- The determination of these **forces** is important in the design of storage tanks, ships, dams and other hydraulic structures.

From Last Lectures

- For Fluid at rest, we knew that the force must be perpendicular to the surface

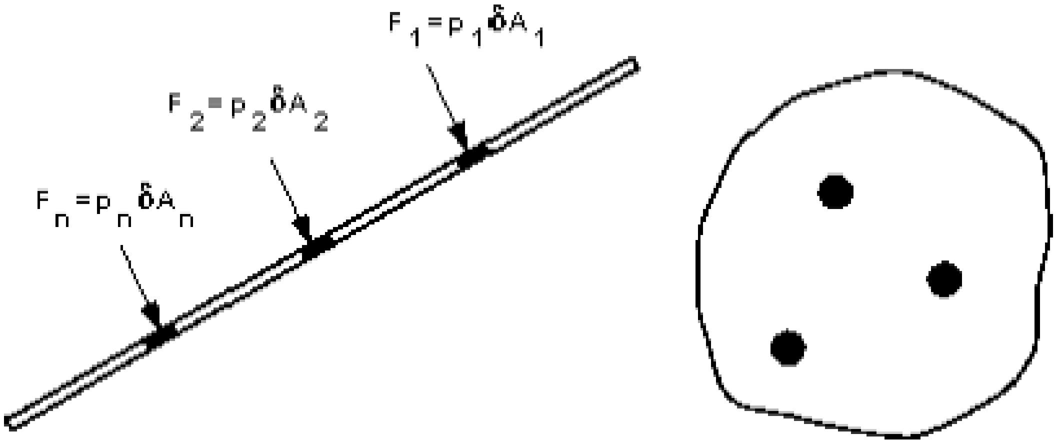
- We knew that the pressure will vary linearly with depth

3.1 Action of Fluid on a Surface

- Pressure is defined as force per unit area. If a pressure *p* acts on a small area *dA* then the force exerted on that area will be:

$$F = pdA$$

- Since the fluid is at rest the force will act at right-angles to the surface.

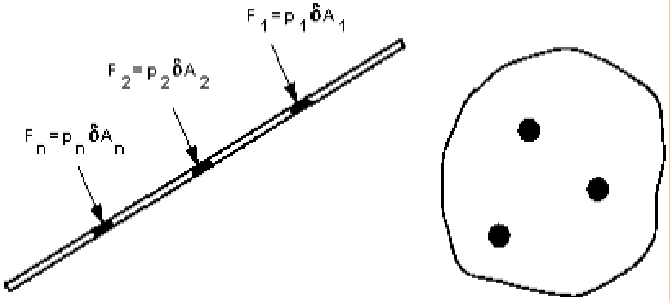


General submerged plane

- Consider the plane surface shown in the figure below.
- The total area is made up of many elemental areas.
- The force on each elemental area is always normal to the surface but, in general, each force is of different magnitude as the pressure usually varies.
- We can find the total or ***resultant*** force, *R*, on the plane by summing up all of the forces on the small elements i.e.

$$R = p_1 dA_1 + p_2 dA_2 + \mathbf{L} + p_n dA_n = \sum pdA$$

If the surface is a **plane** the force can be represented by one single **resultant force**, acting at right-angles to the plane through the **centre of pressure**.



CURVED SUBMERGED SURFACE

- If the surface is curved, each elemental force will be a **different magnitude** and in different **direction** but still normal to the surface of that element.
- The resultant force can be found by **resolving all forces into orthogonal co-ordinate** directions to obtain its magnitude and direction.
- This will always be **less than** the sum of the individual forces,

$$\sum pdA$$

3.2 Resultant Force and Center of Pressure on a Plane Surface under Uniform Pressure

For horizontal plane submerged in a liquid, the pressure, *p*, will be equal at all points of the surface.

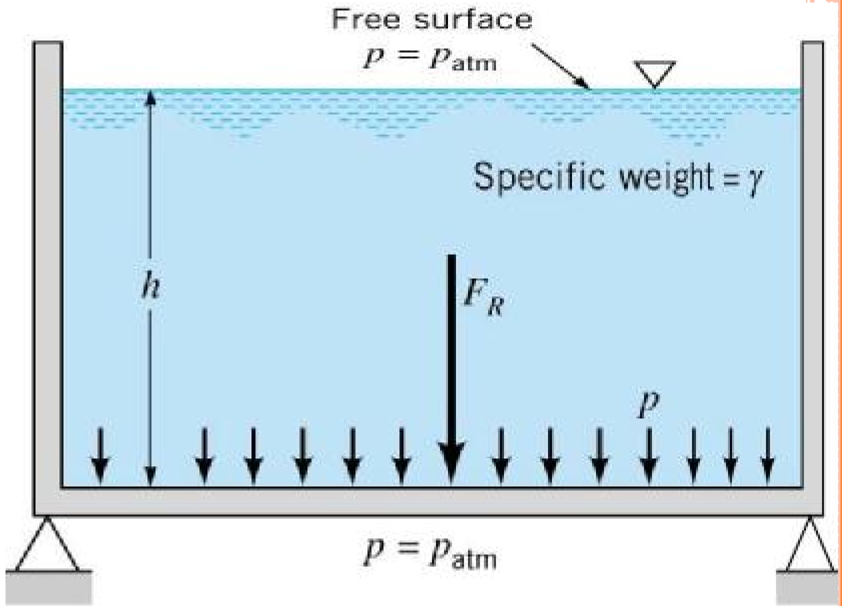
Simplest Case: Tank bottom with a uniform pressure distribution

$$p = rgh = gh$$

Now, the resultant Force:

$$R = pA$$

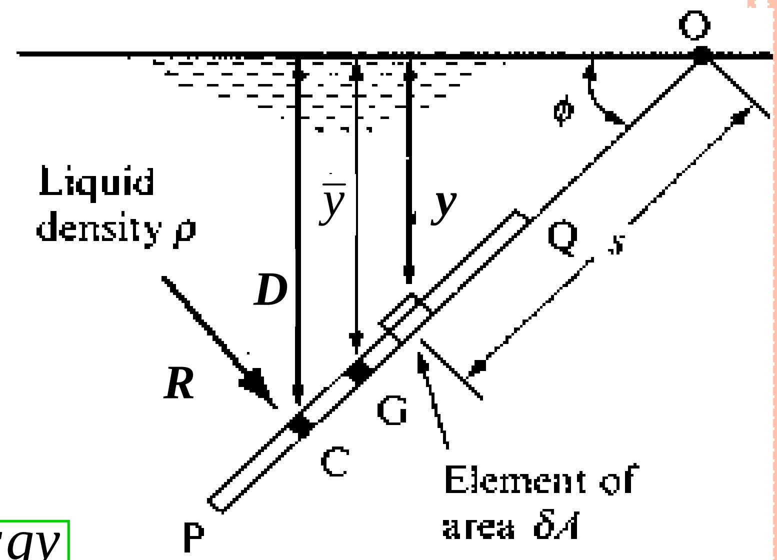
A = area of the plane surface (Tank Bottom)



This force will act vertically downward and the center of pressure will be the **centroid of the surface**

3.3 Resultant Force and Center of Pressure on a Plane Surface Immersed in a Liquid

- Shown a plane surface PQ of an area A submerged in a liquid of density, r, and inclined at an angle f to the free surface.
- Considering one side only, there will be a force due to fluid pressure, acting on each element of area dA,
- The pressure at a distance y below the free surface can be written as: $p = rgy$



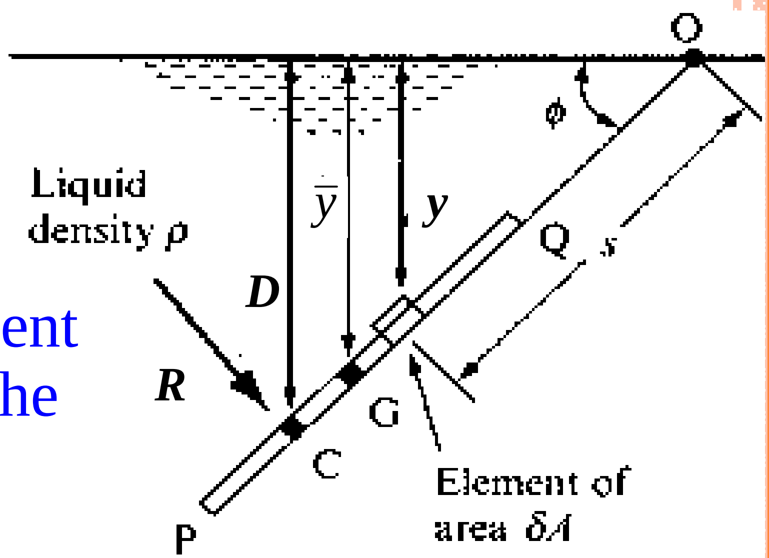
- Force on elemental area dA: $dR = pdA = rgydA$
- The resultant force acting on the plane can be found by summing all the forces on the small element:

$$R = \Sigma pdA = \Sigma rgy\, dA$$

$$R = \Sigma p dA = \Sigma r g y \, dA$$

Assuming that ρ and g are constant

$$R = r g \, \Sigma \, y dA$$



- The quantity $\Sigma y \, dA$ is the first moment of area under the surface *PQ* about the free surface of the liquid

$$\sum y dA = A\bar{y}$$

where

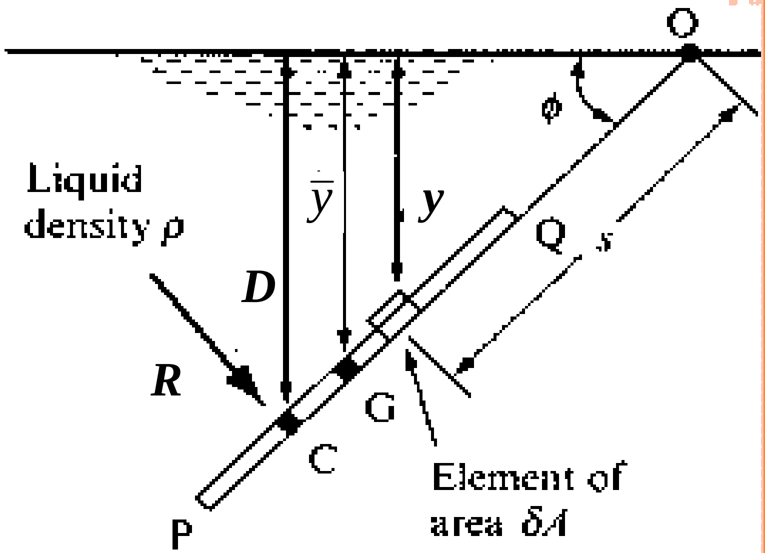
- A = the area of the whole immersed surface and
- $\bar{y}$ = vertical distance from the free surface to the centroid of the area, G, of the immersed surface.

$$\text{Resultant Force} = R = r g A \bar{y}$$

***Remember:** Centroid of the area is defined as the point at which the area would be balanced if suspended from that point.*

$$R = rgA\bar{y}$$

- It may be noted that the resultant force, R, is **independent** of the angle of inclination f as long as the depth of the centroid is unchanged. $\bar{y}$
- The point of application of the resultant force on the submerged area **is called the center of pressure**.



- This resultant force will act perpendicular to the immersed surface **at the center of pressure, *C* at some depth *D* below the free surface.**

- The vertical depth of the center of pressure, ***D***, below the free surface can be found using the following:

$$D = \sin^2 f\left(\frac{I_o}{A\bar{y}}\right) \quad \text{or} \quad D = \sin^2 f\left(\frac{I_G}{A\bar{y}}\right) + \bar{y}$$

where

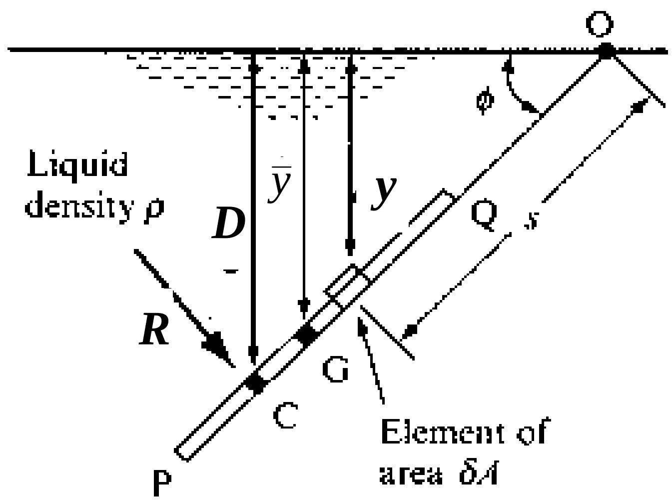
- I_G = second moment of plane area about an axis through its centroid *G* parallel to the free surface
- ***A*** = the area of the whole immersed surface
- $\hat{y} = \bar{y}$ ertical distance from the free surface to the centroid of the area A

The above equation implies that the center of pressure is always below the centroid.

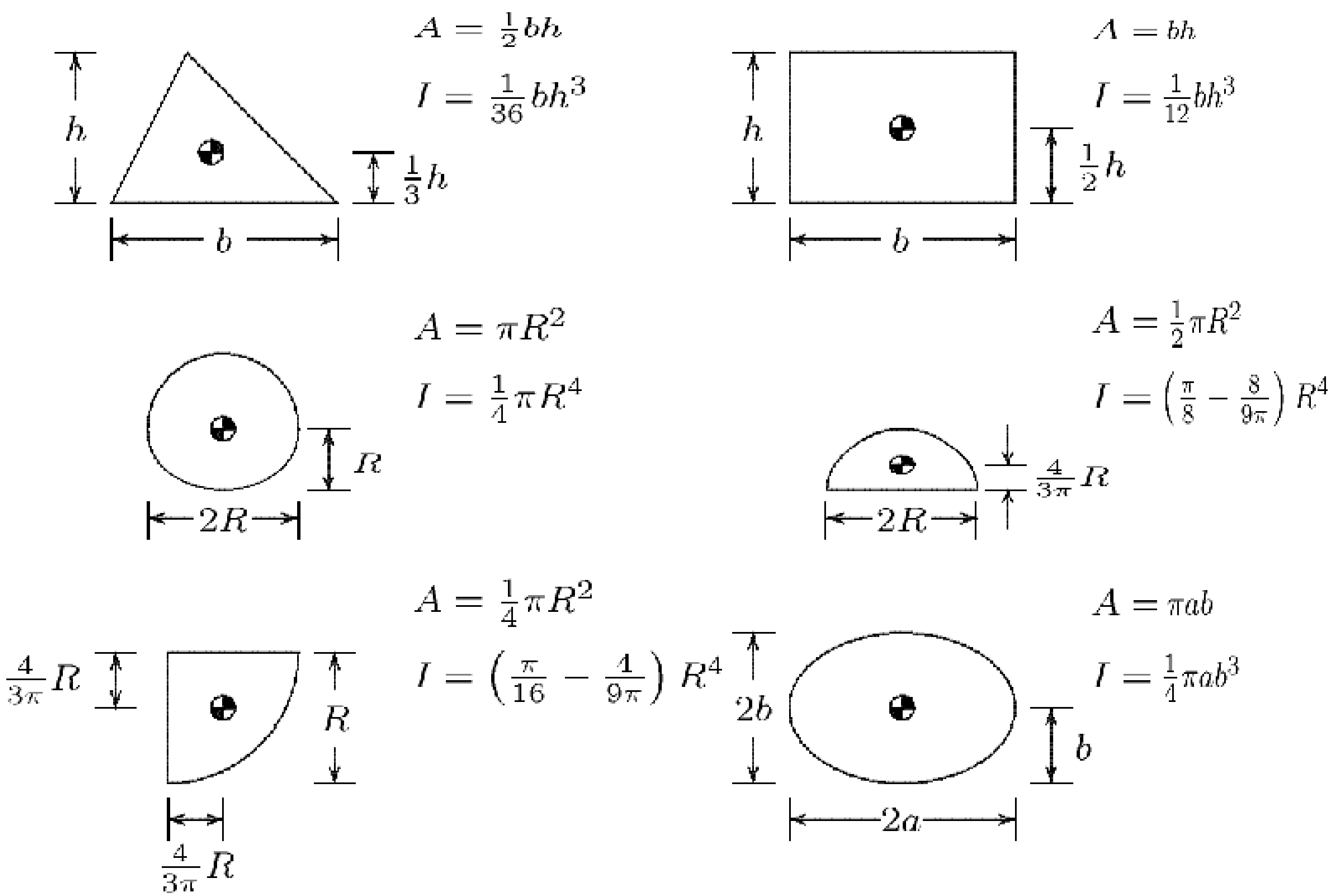
Remember:

Parallel axis theorem

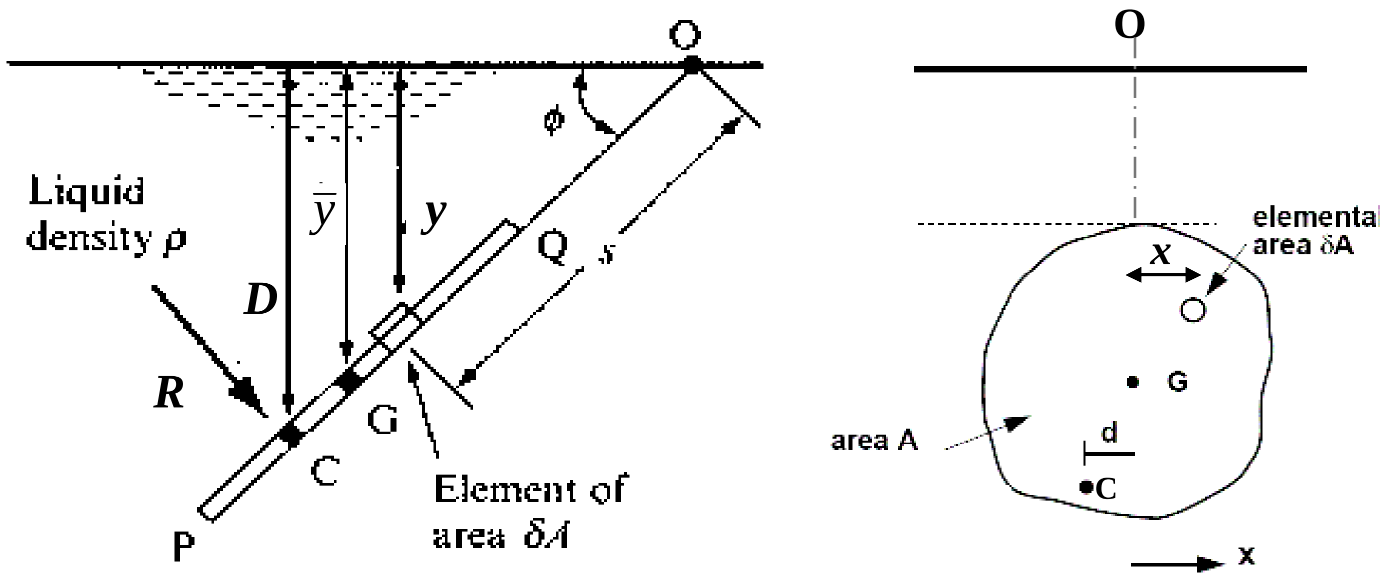
$$I_o = I_G + Ad^2$$



THE SECOND MOMENT OF AREA ABOUT A LINE THROUGH THE CENTROID (I_G) OF SOME COMMON SHAPES.



Lateral position of Centre of Pressure



$R \times d =$ Sum of moments of the forces on all elements about OG

- **If the shape is symmetrical the centre of pressure lies on the line of symmetry.**
- But if it is not symmetrical its position must be found by taking moments about the line OG.

$$= \sum (\rho g y dA) x \qquad \text{But } R = \rho g \bar{y} A$$

$$\Rightarrow \quad d = \frac{\sum yxdA}{A\bar{y}}$$

Example 3.1 page 65

HW: Example 3.2 page 67

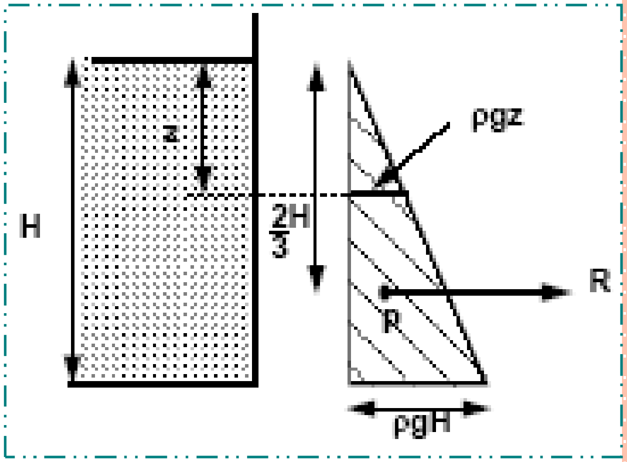
3.4 PRESSURE DIAGRAMS

- For vertical walls of constant width it is possible to find the resultant force and centre of pressure graphically using a ***pressure diagram***.
- We know the relationship between pressure and depth:

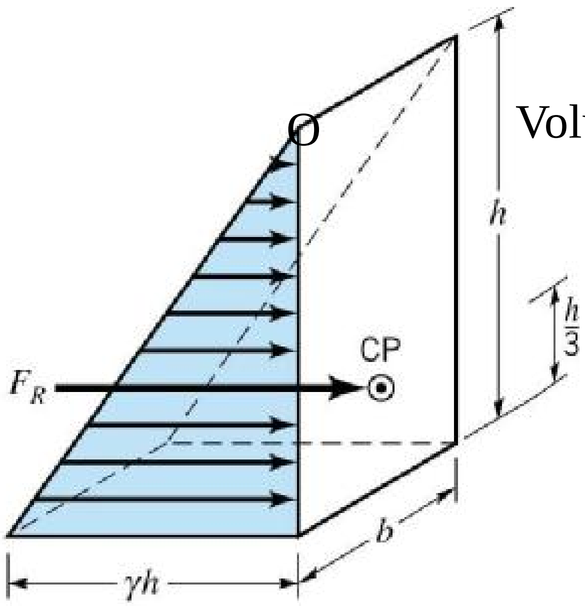
$$p = rgz$$

Consider the tank shown:

- We can draw the shown diagram **(graphical representation of the (gauge) pressure change with depth on one of the vertical walls.)**
- It increases linearly from zero at the surface by $p = rgy$, to a maximum at the base of $P = rgH$.
- This is know as a **pressure diagram.**



Pressure Diagram: A graphical interpretation of the forces due to a fluid acting on a plane area. The "volume" of fluid acting on the wall is the pressure diagram and equals the resultant force acting on the wall.



Resultant Force:

$$F_R = \text{volume}$$

$$F_R = \frac{1}{2}(rgh)(h)(b)$$

$$F_R = \frac{1}{2}(rgh^2)b$$

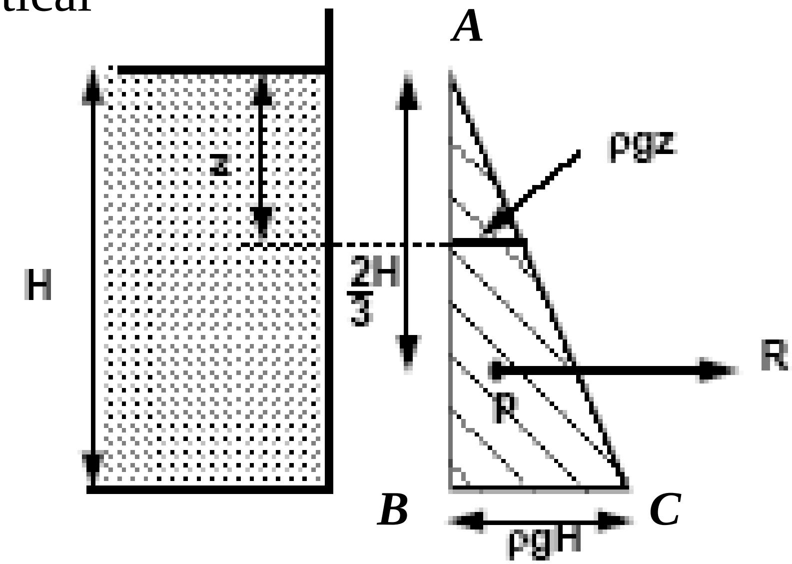
Location of the Resultant Force, C:

The location is at the centroid of the volume of the pressure diagram.

Center of Pressure:

$$\left(\frac{b}{2}, \frac{2h}{3}\right)$$

- The area of this triangle represents the **resultant force per unit width** on the vertical wall, (N/m). So:



$$Area = \frac{1}{2} AB \times BC = \frac{1}{2} H \rho g H = \frac{1}{2} \rho g H^2$$

Resultant force per unit width

$$R = \frac{1}{2} \rho g H^2 \quad \text{(N/m)}$$

§ This force acts through the centroid of the pressure diagram. For a triangle, the centroid is located at 2/3 its height, thus the resultant force acts at a depth of 2/3 *H* from the surface.

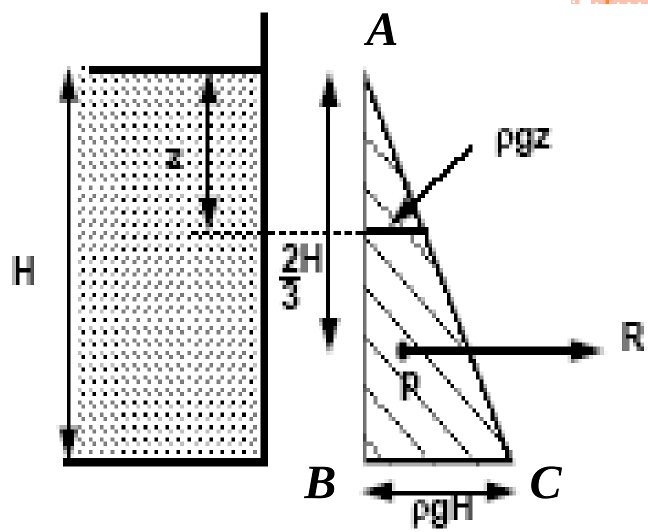
§ The total resultant force can be obtained by multiplying the above equation with the width of the surface, *B*.

$$R = \frac{1}{2} \rho g H^2 B \quad \text{(N)}$$

THIS CAN BE CHECKED AGAINST THE PREVIOUS METHOD:

$$R = \rho g A \bar{y}$$

$$R = \rho g (H \times 1) \frac{H}{2} = \frac{1}{2} \rho g H^2$$ **R**

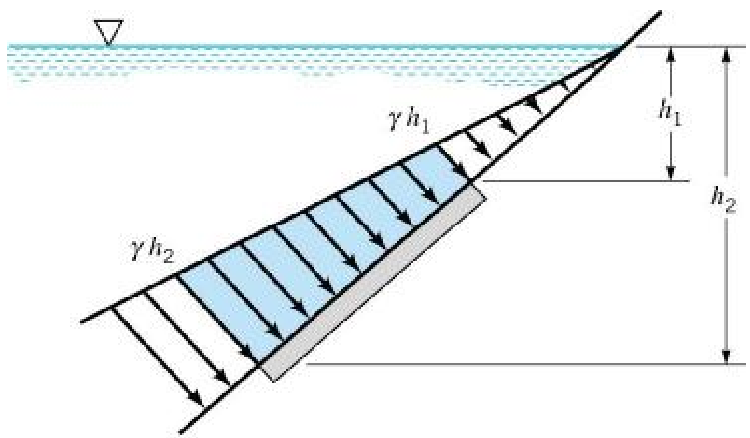


$$D = \sin^2 \phi \left(\frac{I_o}{A\bar{y}} \right)$$ ***or*** $$D = \sin^2 \phi \left(\frac{I_g}{A\bar{y}} \right) + \bar{y}$$

$\phi = 90^o$, $\sin \phi = 1$, $\bar{y} = H/2$, $Ig = H^3/12$

$$D = \left(\frac{H^3/12}{(H \times 1) \times (H/2)} \right) + \frac{H}{2} = \frac{H}{6} + \frac{H}{2} = \frac{4H}{6} = \frac{2H}{3}$$ **R**

- If the plane surface is inclined and submerged below the surface, the pressure diagram is drawn perpendicular to the immersed surface and will be straight line extending from $p=0$ at the free surface to $p=\rho g h_2$ at depth h_2 .
- As the immersed surface does not extend to the free surface, the resultant force *R* is represented by the shaded area, instead of the whole triangle, and acts through the centroid *P* of this shaded area.



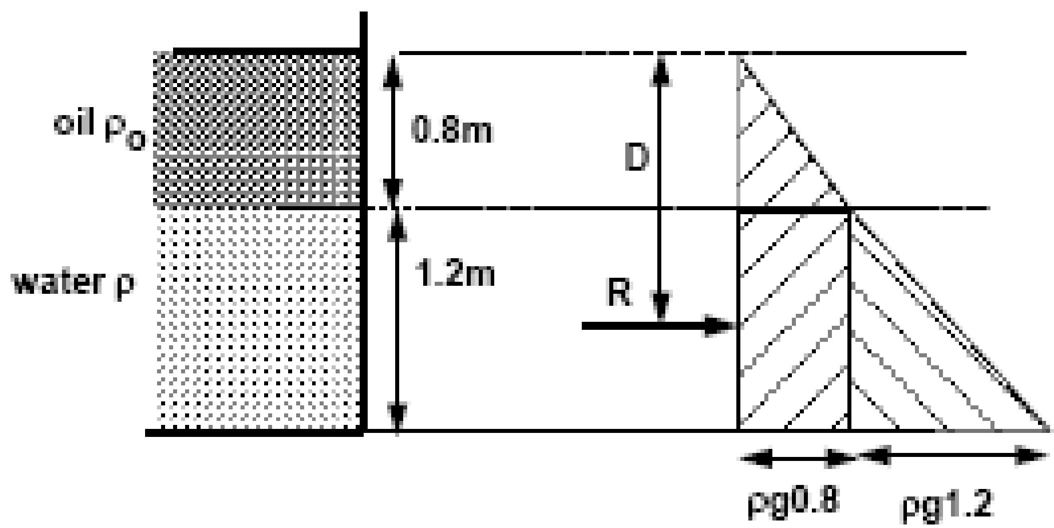
Note

- More complex pressure diagrams can be draw for non-rectangular or non-vertical planes but it is usually far easier to use the moments method.

- The same pressure diagram technique can be used when combinations of liquids are held in tanks (e.g. oil floating on water) with position of action found by taking moments of the individual resultant forces for each fluid.

For example:

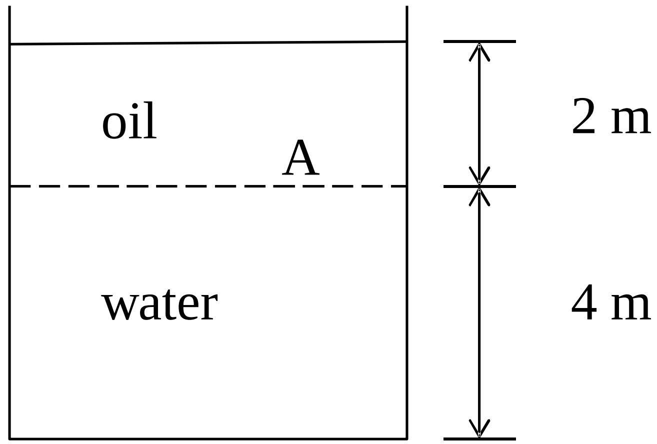
- Find the position and magnitude of the resultant force on this vertical wall of a tank which has oil floating on water as shown.



EXAMPLES

A 6-m deep tank contains 4 m of water and 2-m of oil as shown in the diagram below.

- Determine the pressure at point A and at the bottom of the tank.
- Draw the pressure diagram.



$r_{water} = 1000\ kg/m^3$

SG of oil = 0.98

Solution:

Pressure at oil water interface (P_A)

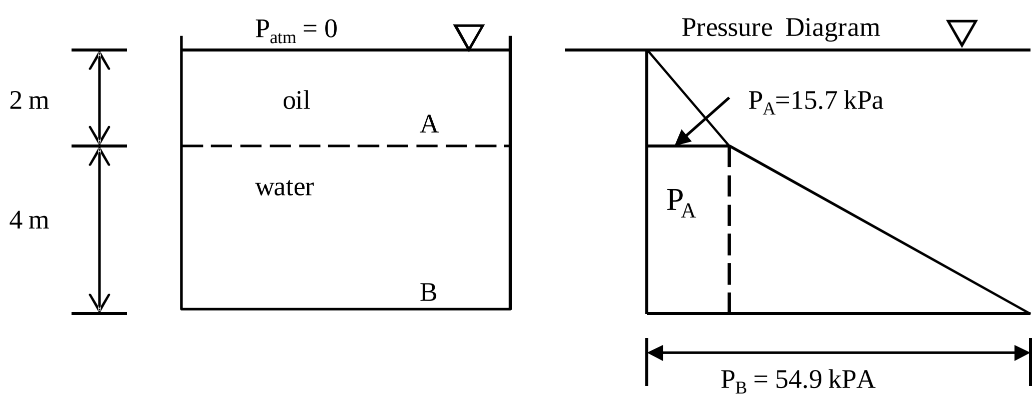
$$P_A = P_{atm} + P_{oil} \text{ (due to 2 m of oil)}$$
$$= 0 + r_{oil} g h_{oil} = 0 + 0.98 \text{ x } 1000 \text{ x } 9.81 \text{ x } 2$$
$$= 15696 \text{ Pa}$$

$$\boldsymbol{P_A = 15.7 \text{ kPa (gauge)}}$$

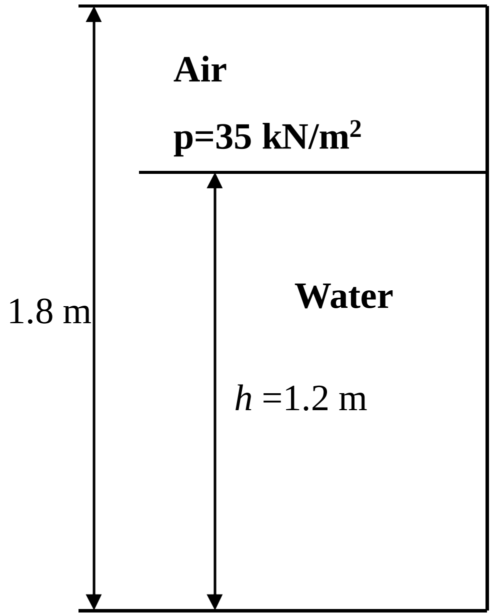
Pressure at the bottom of the tank;

$$P_B = P_A + r_{water} g h_{water}$$
$$P_B = 15.7\text{x}1000 + 1000 \text{ x } 9.81 \text{ x } 4$$
$$= 54940 \text{ Pa}$$

$$\boldsymbol{P_B = 54.9 \text{ kPa (gauge)}}$$



EXAMPLES 3.3 PAGE 70



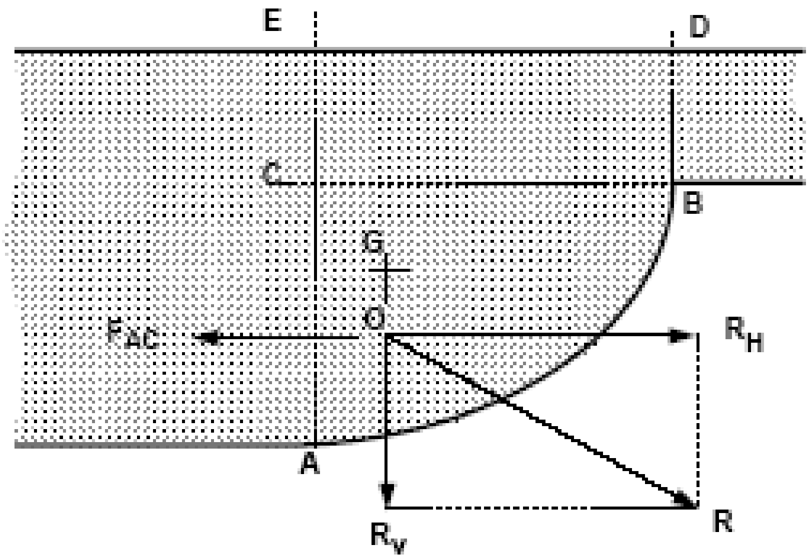
Determine the resultant force on this wall and the height of the center of pressure above the base

The length of the wall of tank is 3m

3.5 FORCE ON A CURVED SURFACE DUE TO HYDROSTATIC PRESSURE

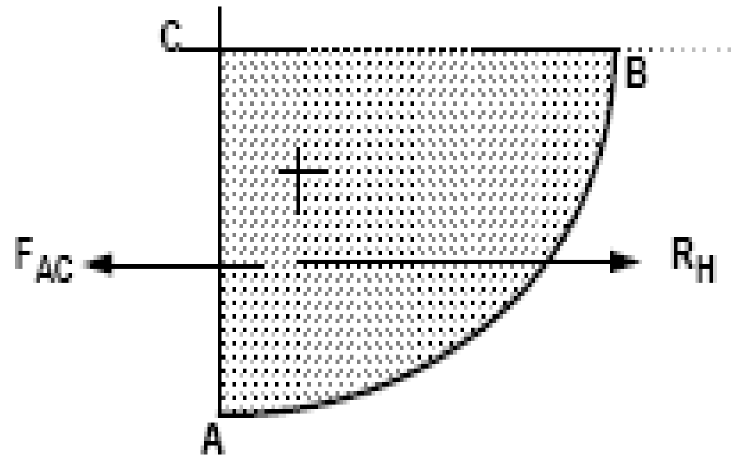
- Many surfaces in dams, pumps, pipes or tanks are curved
- General theory of plane surfaces does not apply to curved surfaces
- As stated above, if the surface is curved the **forces on each element of the surface will not be parallel** and must be combined using some vectorial method.
- It is most straightforward to calculate the **horizontal** and **vertical** components.
- Then combine these forces to obtain the resultant force and its direction.

- In the diagram below liquid is resting on top of a curved base.
- The element of fluid ABC is equilibrium (as the fluid is at rest).



<u>Consider the Horizontal forces</u>

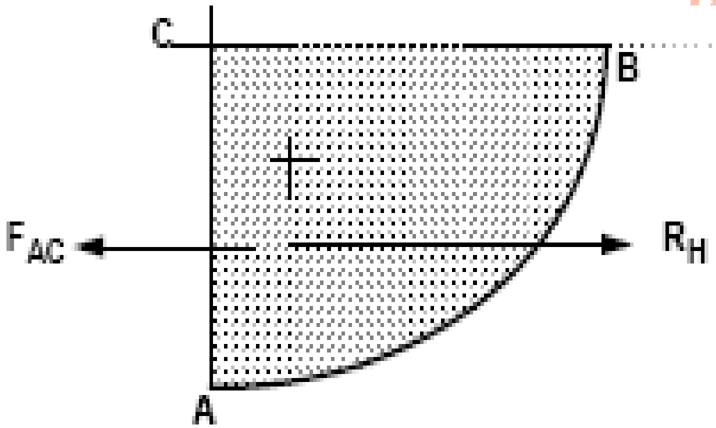
- The sum of the horizontal forces is zero.
- F_{AC}, must be equal and opposite to R_H



The resultant horizontal force of a fluid above a curved surface is:
R_H = Resultant force on the projection of the curved surface onto a vertical plane.

We know that:

- The force on a vertical plane must act horizontally (as it acts normal to the plane).
- R_H must act through the same point.



So we can say

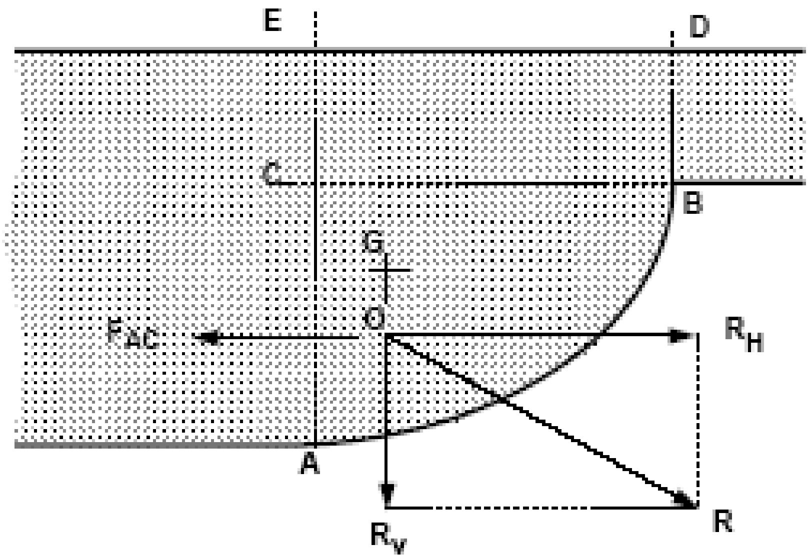
> R_H acts horizontally through the centre of pressure of the projection of the curved surface onto a vertical plane.

Note:

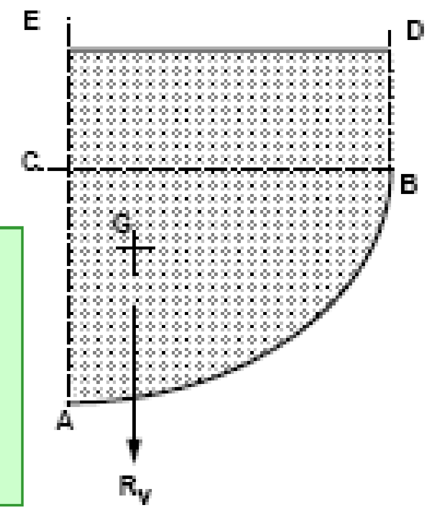
We can use **the pressure diagram method** to calculate the position and magnitude of the resultant horizontal force on a curved surface.

Consider the Vertical forces

- The sum of the vertical forces is zero.
- There are no shear force on the vertical edges, so the vertical component can only be due to the weight of the fluid.



The resultant vertical force of a fluid above a curved surface is:
R_V = Weight of fluid directly above the curved surface.



It will act vertically downward through the centre of gravity of the mass of fluid.

Resultant force

- The overall resultant force is found by combining the vertical and horizontal components vectorialy:

$$R = \sqrt{R_H^2 + R_V^2}$$

- And acts through *O* at an angle of *u*.
- The angle the resultant force makes to the horizontal is:

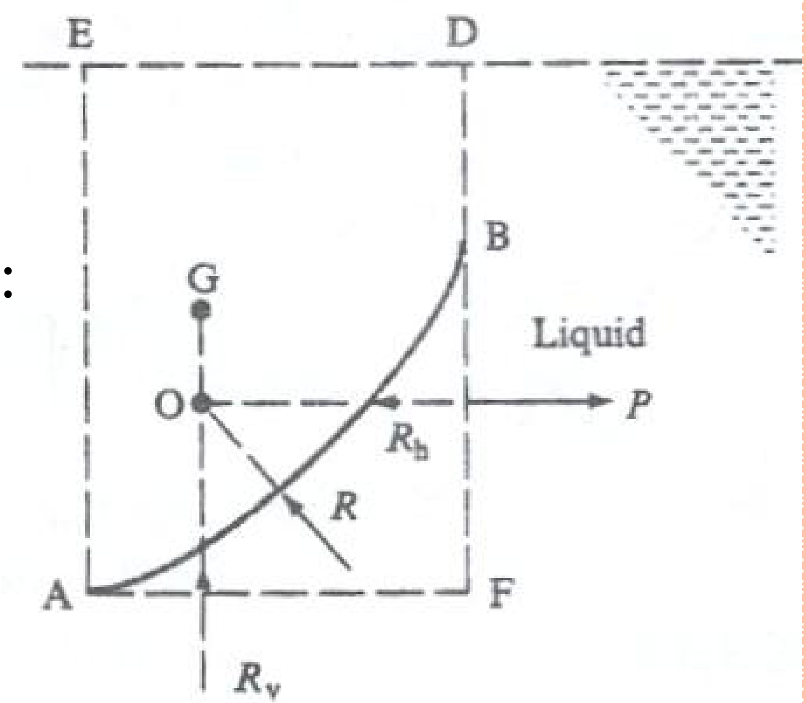
$$q = \tan^{-1}\left(\frac{R_V}{R_H}\right)$$

- The position of *O* is the point of intersection of the horizontal line of action of R_H and the vertical line of action of R_V .

What are the forces if the fluid is ***below*** the curved surface?

<u>Consider the Horizontal forces</u>

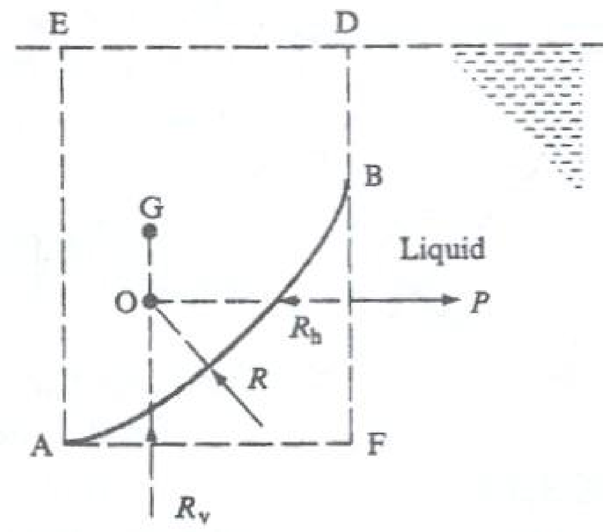
- There are two horizontal forces on the element which is in equilibrium:
- The horizontal reaction force R_H
- The pressure force on the vertical plane *FB*.



The resultant horizontal force of a fluid below a curved surface is:
R_H = Resultant force on the projection of the curved surface onto a vertical plane.

Consider the Vertical forces

- If the curved surface were removed and the area it were replaced by the fluid, the whole system would be in equilibrium.
- Thus the force required by the curved surface to maintain equilibrium is equal to that force which the fluid above the surface would exert - i.e. the weight of the fluid.

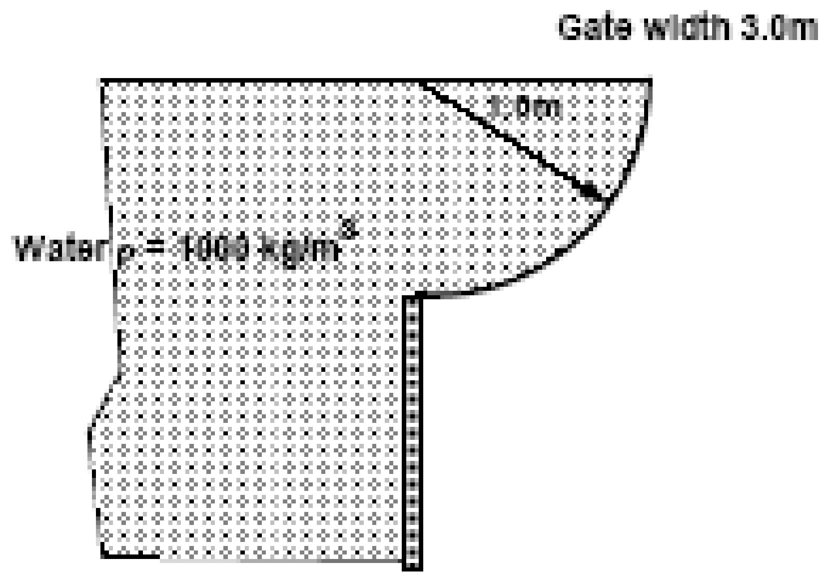


What vertical force would keep this in equilibrium?

The resultant vertical force of a fluid below a curved surface is:
R_V = Weight of the *imaginary* volume of fluid vertically above the curved surface.

EXAMPLES

Find the magnitude and direction of the resultant force of water on a quadrant gate as shown below.

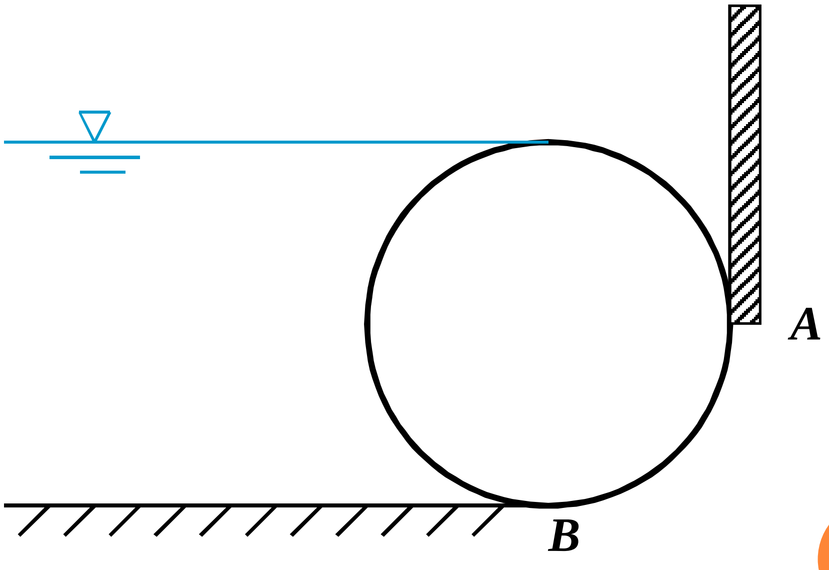


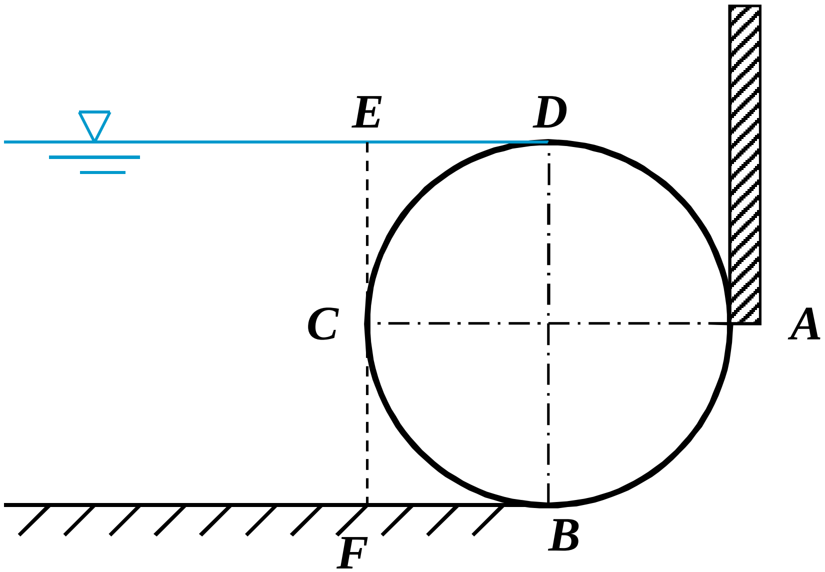
EXAMPLES

A 2m long cylinder lies as shown in the figure (diameter = 2m also) , holding back oil of relative density 0.8. If the cylinder has a mass of 3250 kg.

Find:

- the reaction at *A*
- the reaction at *B*




E
D
C
A
F
B

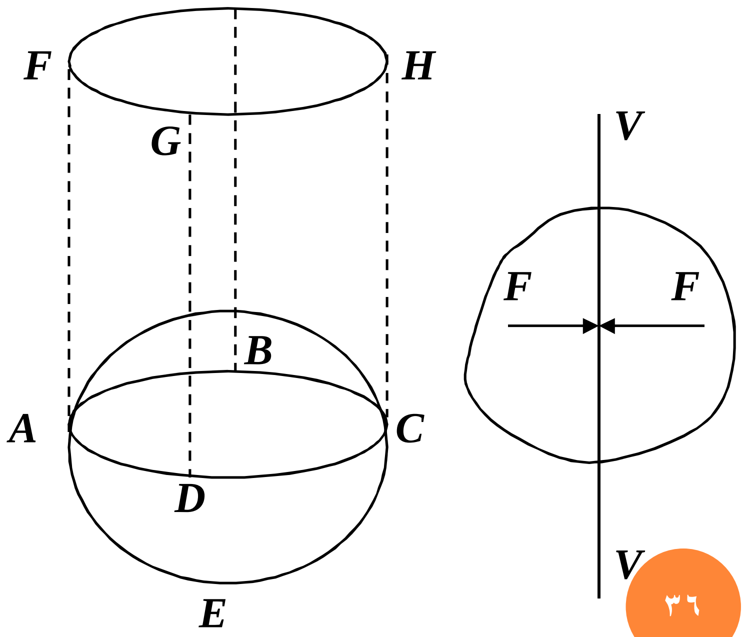
HW: Example 3.4 page 72

3.6 BUOYANCY

- When a body is submerged or floating in a static fluid, the resultant force exerted on it by the fluid is called the buoyancy force.
- Buoyant force can be defined as the resultant fluid force which acts on a fully submerged or floating body.

- Consider the vertical plane *VV*
- The projected area of each of the two sides on this plane will be equal.
- The horizontal forces *F* will be equal and opposite.

No resultant horizontal force on the body due to the pressure of the surrounding fluid



In the vertical direction:

- A force is exerted by the fluid on the immersed body called the buoyancy or upthrust force.

- Upthrust on the body = Upward force on lower surface ADEC

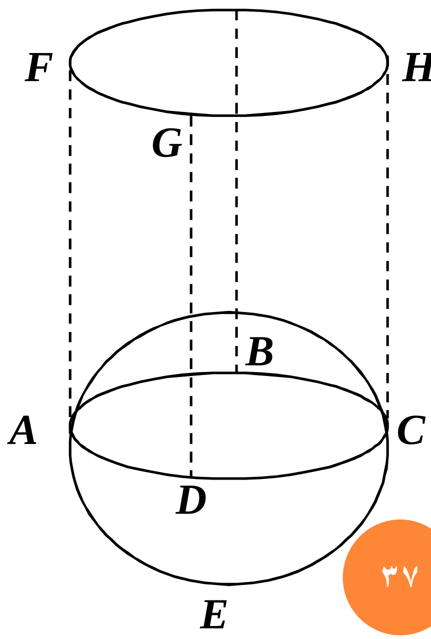
  - Downward force on the upper surface ABCD

  = Weight of volume of fluid AECDGFH

  - Weight of volume of fluid ABCDGFH

  = Weight of volume of fluid ABCDE

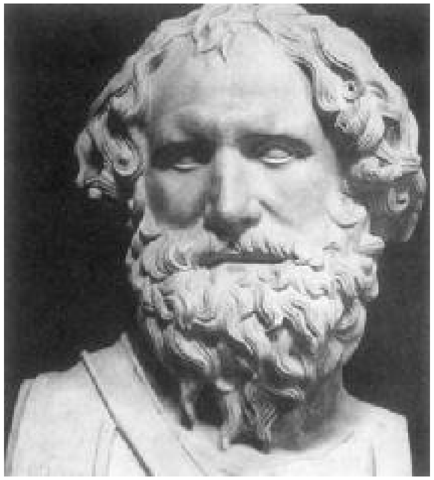
Upthrust on the body = Weight of fluid displaced by the body



- This force will act vertically upward through the centroid of the volume of fluid displaced, known as **the centre of buoyancy**

Archimedes' principle

Buoyancy: Archimedes' Principle



Archimedes (287-212 BC)

Archimedes' Principle states that the buoyant force has a magnitude equal to the weight of the fluid displaced by the body and is directed vertically upward.

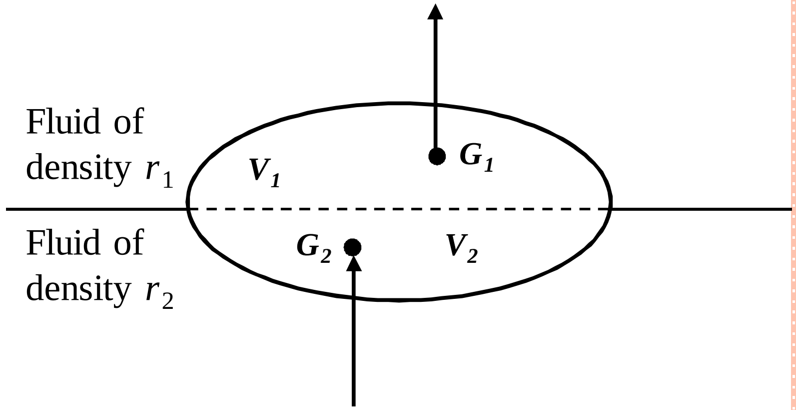
$$R = F_b = r_{Fluid}\, g V_{Displaced\ by\ Body}$$

If a body is immersed in two different fluid as shown

- Upthrust on upper part:

$$R_1 = r_1 g V_1$$

Acting through G_1, the centroid of V_1



- Upthrust on lower part:

$$R_2 = r_2 g V_2$$

Acting through G_2, the centroid of V_2

- Total Upthrust on the body:

$$R = R_1 + R_2 = r_1 g V_1 + r_2 g V_2$$

EXAMPLE 3.5 PAGE 75 (BOOK)

A rectangular pontoon has a width *B* of 6m, a length *L* of 12m, and a draught *D* of 1.5m in fresh water (*r* =1000 kg/m3).

Calculate:

a) The weight of the pontoon.

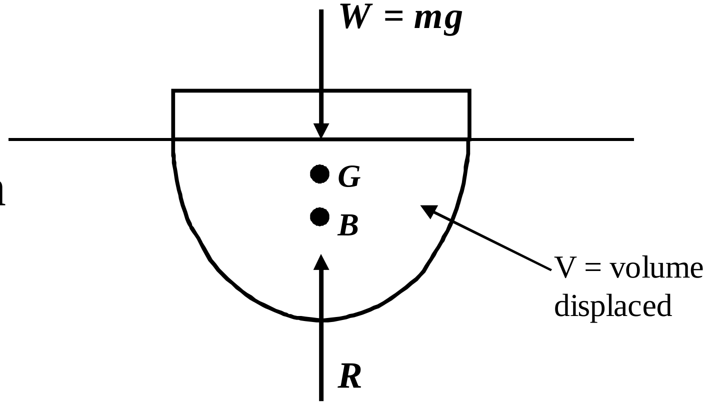
b) Its draught in sea water (*r* =1025 kg/m3)

c) The load that can be supported by the pontoon in fresh water if the maximum draught permissible is 2 m

3.7 EQUILIBRIUM OF FLOATING BODIES

When the body (Ship) floats in vertical equilibrium in a liquid, the forces presented are:

1. The upthrust force *R* acting through the center of buoyancy *B*.
2. The weight of the body *W* acting through its center of gravity *G*.

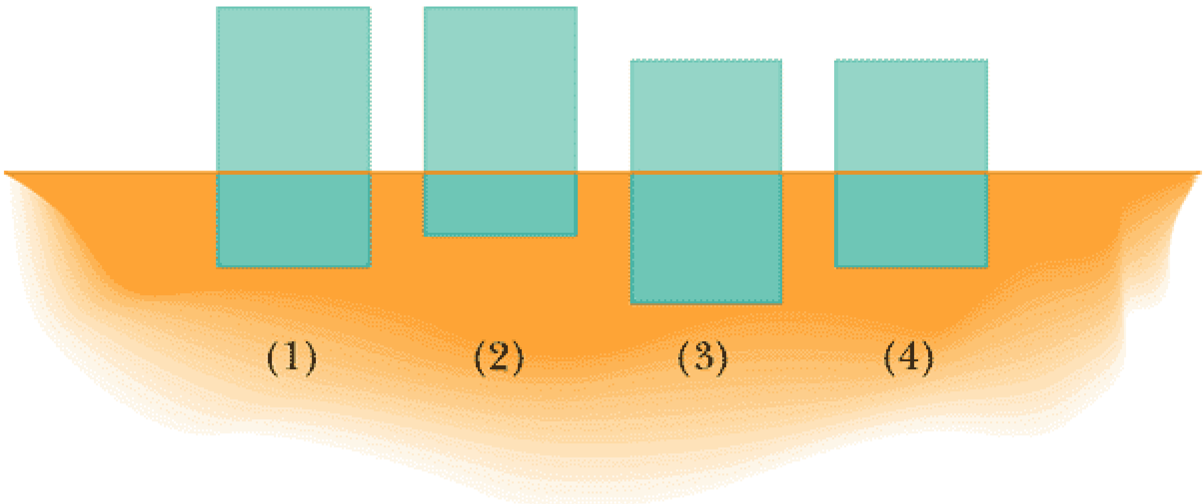


For equilibrium:

1. *R* and *W* must be equal.
2. *R* and *W* must act in the same straight line.

$$W = R$$

$$mg = rVg \Rightarrow V = \frac{m}{r}$$



Four solid objects are floating in corn syrup.
Rank the objects according to their density, greatest first.

The equilibrium of a body may be:

- Stable
- Unstable
- Neutral

- Depending upon whether, when given a small displacement, it tends to return to the equilibrium position, move further from it or remain in the displaced position.

For <u>floating</u> body (such as ship) stability is of major importance.

3.8 STABILITY OF A SUBMERGED BODY

Remember

Stable Equilibrium: if when displaced returns to equilibrium position.

Unstable Equilibrium: if when displaced it returns to a new equilibrium position.

- The upthrust force *R* acting through the center of buoyancy *B*.
- The weight of the body *W* acting through its center of gravity of the body *G*.

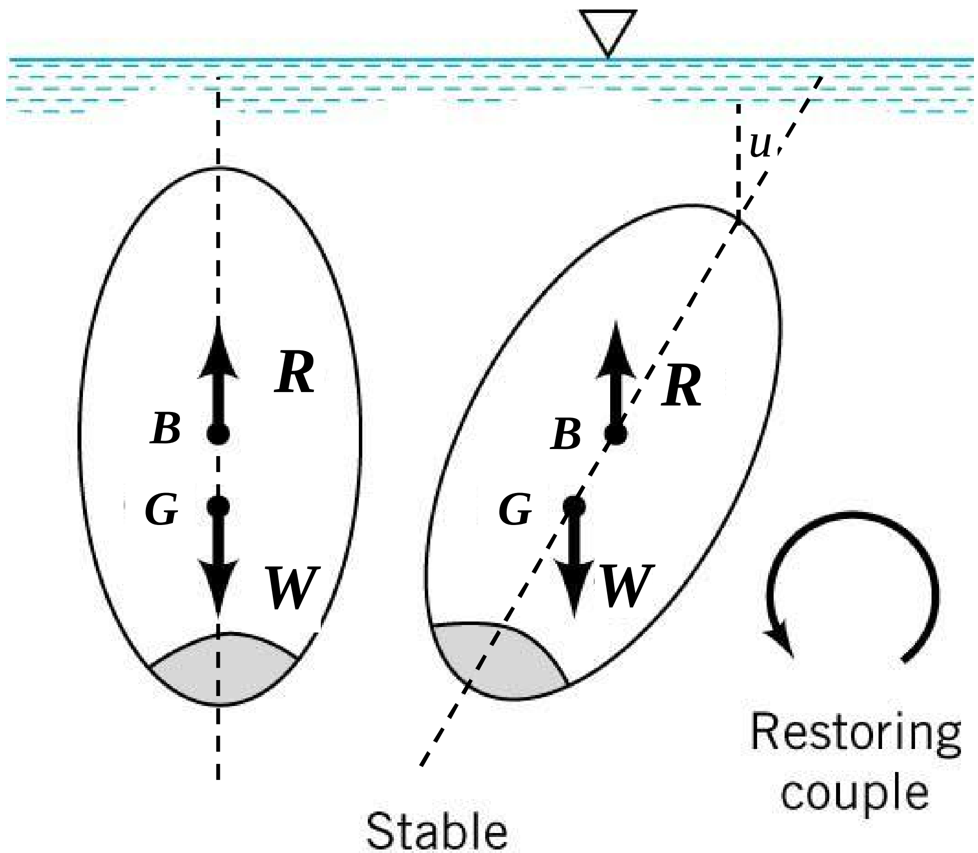
Whatever the orientation of the body, these two points will remain in the same positions relative to the body.

Note: as the body is totally immersed, the shape of the displaced fluid is not altered when the body is tilted and so the centre of buoyancy remains unchanged relative to the body.

It can be seen that: A small angular displacement u from the equilibrium position will generate a moment $W.BG.u$.

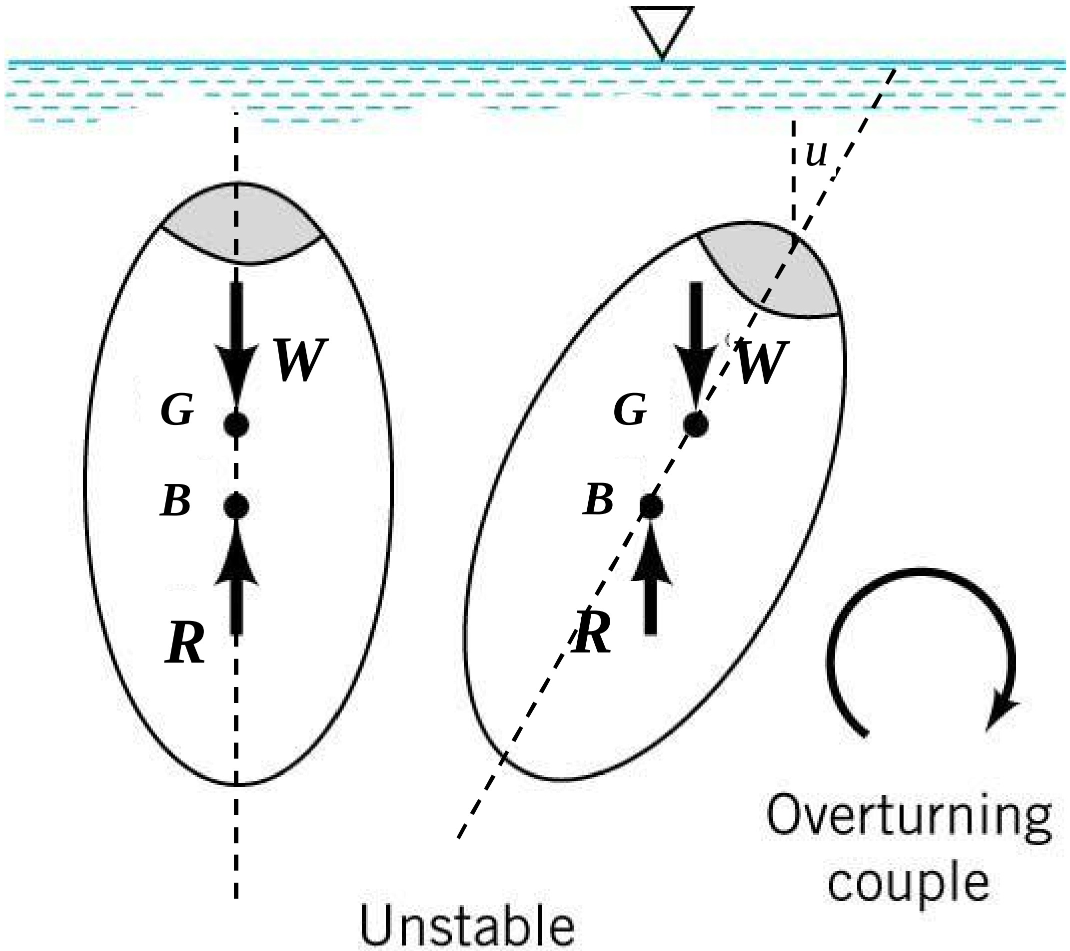
Stable

If ***G is below B***, this will generate **righting moment** and the body will tend to return to its equilibrium position

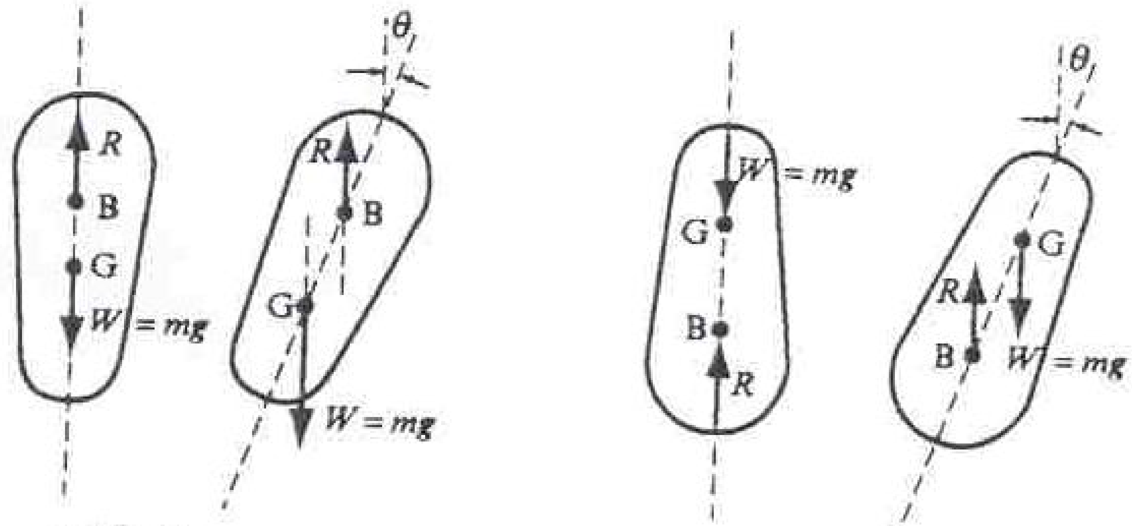


Unstable

If ***G is above B***, this will be a **overturning moment** and the body is unstable



Summary



(a) Stable

(b) Unstable

3.9 Stability of Floating Bodies

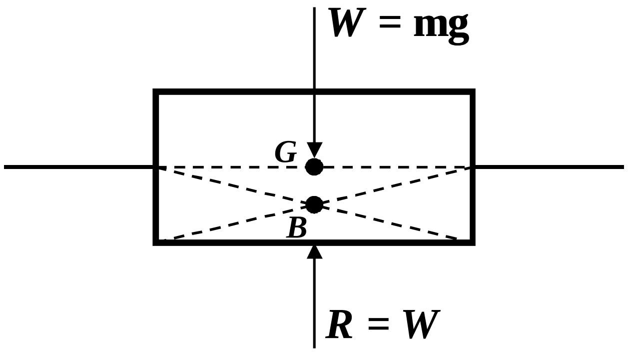
- It is **slightly more complicated** as the location of the center of buoyancy can change:
- The volume of liquid remains unchanged since $R=W$, but the **shape of this volume changes** and therefore its centre of buoyancy.
- When the is displaced through an angle u the center of buoyancy move from B to B_1 and a turning moment is produced.
- Let M (**metacentre**) is the point at which the line of action of the upthrust R cuts the original vertical line through G, then:

$$\text{Moment generated} = W \times GM \times q$$

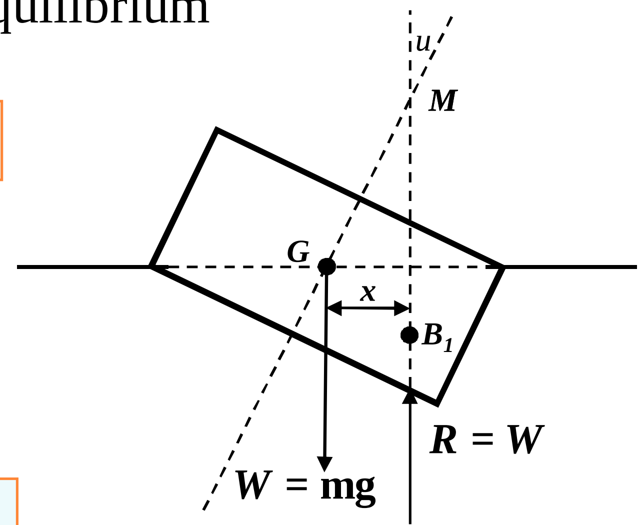
- GM known as the **metacentric height**

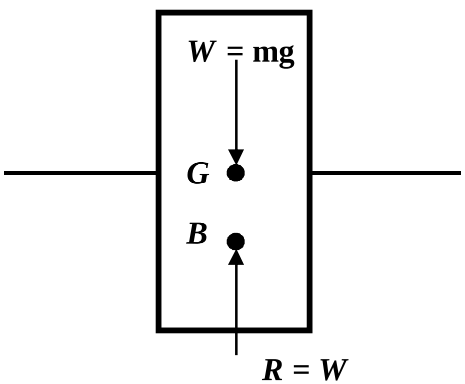
It can be seen that:

- If *M* lies above *G*, a righting moment is produced, equilibrium is stable and *GM* is regarded as positive.
- If *M* lies below *G*, an overturning moment is produced, equilibrium is unstable and *GM* is regarded as negative.
- If *M* coincides with *G*, the body is in neutral equilibrium

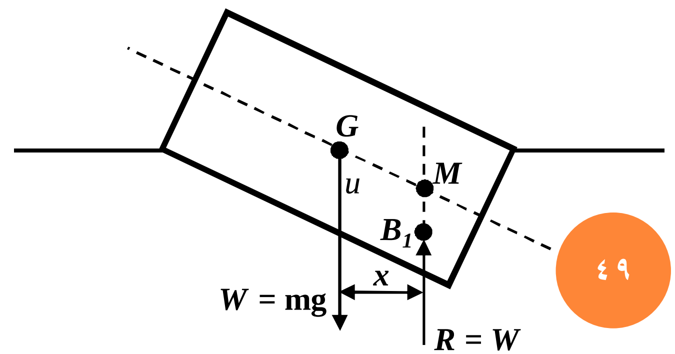


Stable

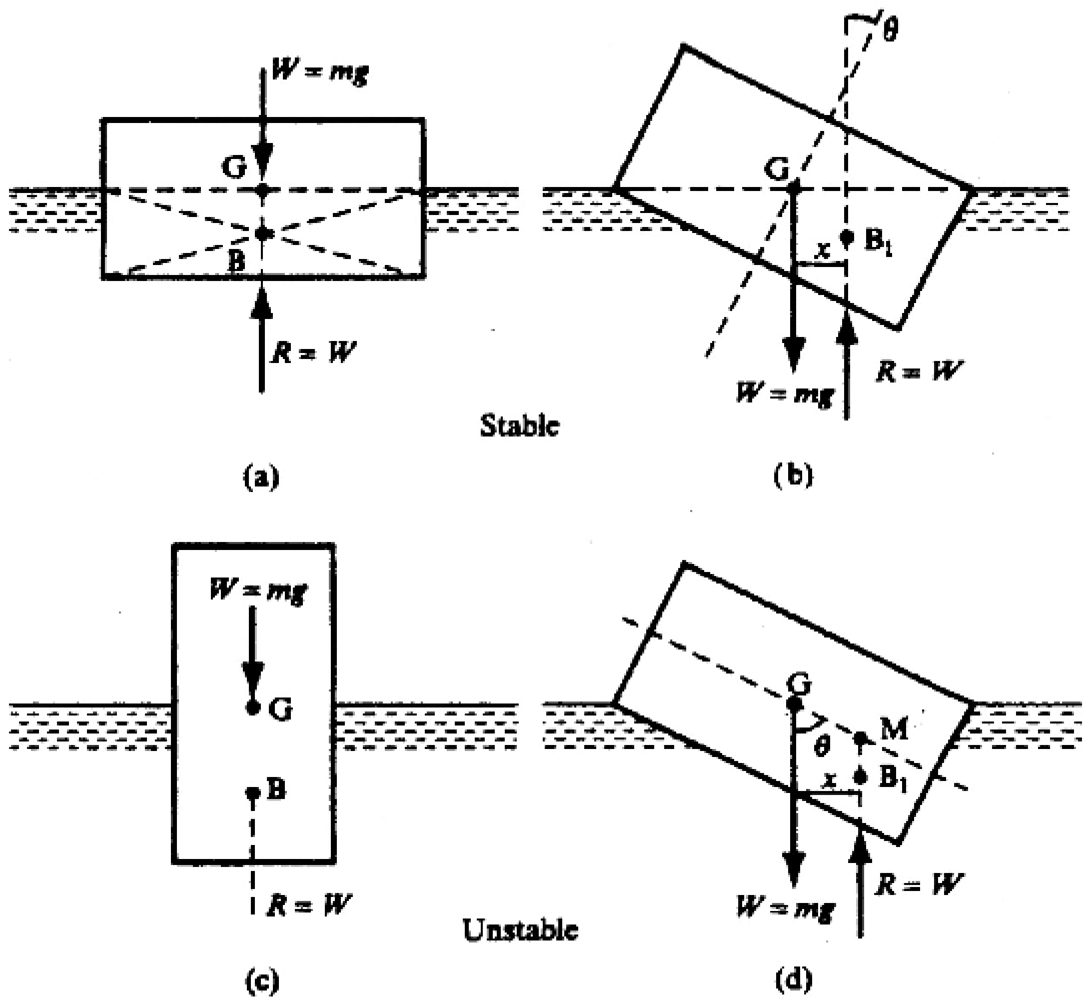




Unstable



Summary


W = mg
G
B
R = W
θ
G
x
B1
W = mg
R = W
Stable
(a)
(b)
W = mg
G
B
R = W
G
θ
M
x
B1
W = mg
R = W
Unstable
(c)
(d)

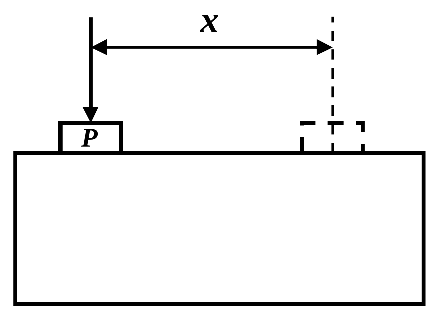
3.10 DETERMINATION OF METACENTRIC HEIGHT

- The metacentric height of a vessel can be measured if the angle of tilt u caused by moving a load P a known distance x across the deck is measured.

The overturning moment due to movement of load P = Righting moment

$$W \times GM \times q = Px$$

$$GM = \frac{Px}{Wq}$$



Where:

W: is the weight of the vessel including P

3.11 Determination of the Position of the Metacentre Relative to the Centre of Buoyancy

When a vessel is tilted trough a small angle *u* then:

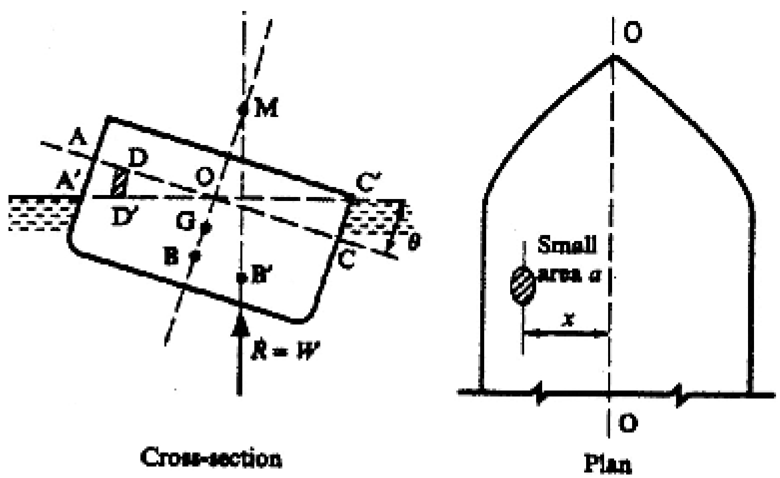
- The centre of buoyancy *B* will move to *B'*.
- The total weight displaced remains unchanged.

The distanced *BB'* can be calculated as follows:

$$BB^{'} = \frac{qI}{V}$$

The metacentric radius *BM* :

$$BM = \frac{BB^{'}}{q} = \frac{I}{V}$$



Where:

u : tilted angle of the vessel

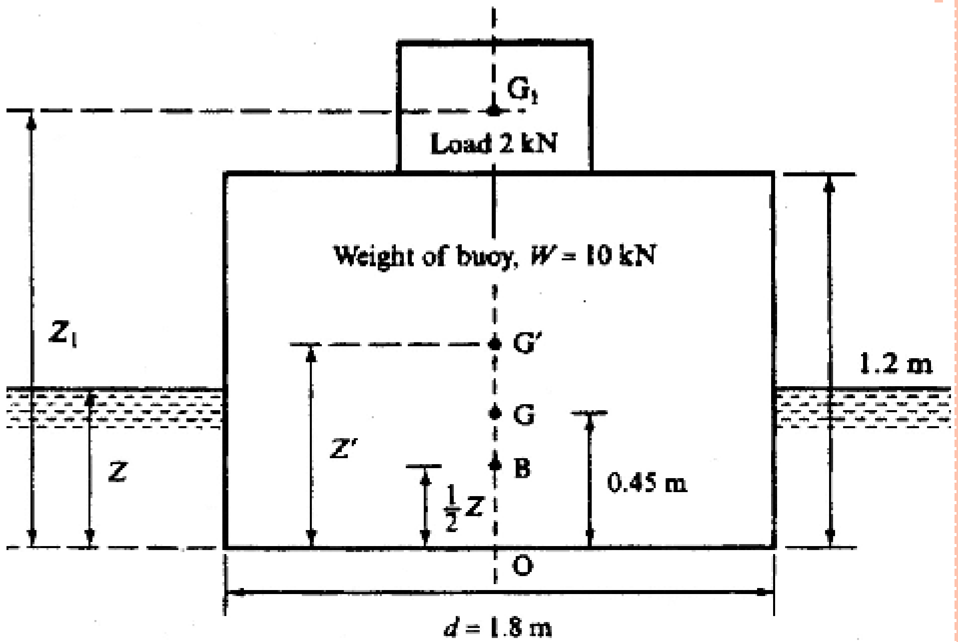
I : second moment of area of the waterline plane about *OO*

V : Volume of liquid displaced

EXAMPLE 3.6 PAGE 80 (BOOK)

A cylindrical buoy 1.8-m in diameter, 1.2-m high and weighting 10-kN floats in salt water of density 1025-kg/m^3.

Its center of gravity is 0.45-m from the bottom. If a load of 2-kN is placed on the top.



Find the ***maximum height*** of the center of gravity of this load above the bottom if the buoy is to remain in stable equilibrium

EXAMPLE

- A circular door having a diameter of 4 m is positioned at the inclined wall as shown in Fig. which forms part of a large water tank. The door is mounted on a shaft which acts to close the door by rotating it and the door is restrained by a stopper. If the depth of the water is 10 m at the level of the shaft, Calculate:

  (a) Magnitude of the hydrostatic force acting on the door and its centre of pressure,

  (b) The moment required by the shaft to open the door.

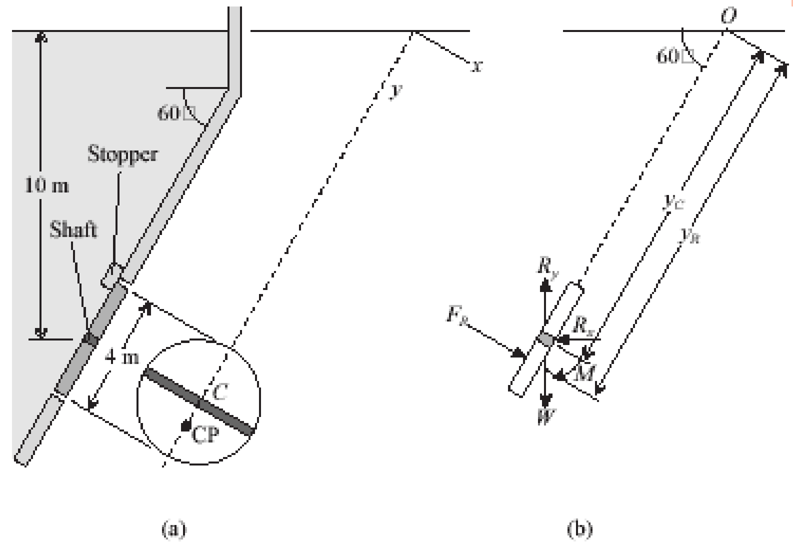


Figure E2.3

- Use ρ_{water} = 998 kg/m^3 and g = 9.81 m/s^2.

§(a) The magnitude of the hydrostatic force F_R is

$$F_R = \rho g y A$$

$$= (998)(9.81)(10)\ [\ ¼\ \pi(4)^2]$$

$$= 1.230 \times 10^6 \text{ N}$$

$$= \underline{1.23 \text{ MN}}$$

§For the coordinate system shown in Figure E2.3(b), since circle is a symmetrical shape, $I_{xy} = 0$, then $x_R = 0$. For y coordinate,

$$y_R = \frac{I_{xx}}{y_C A} + y_C = \frac{¼\ \pi R^4}{\pi\ y_C R^2} + y_C$$

$$= \frac{¼\ \pi\ (2)^4}{\pi\ (10/\sin 60°)(2)^2} + \frac{10}{\sin 60°}$$

$$= \underline{11.6 \text{ m}}$$

§Use moment equilibrium $\Sigma M - 0$ about the shaft axis. With reference to Figure E2.3(b), the moment M required to open the door is:

$$M = F_R\ (\ y_R - y_C\ )$$

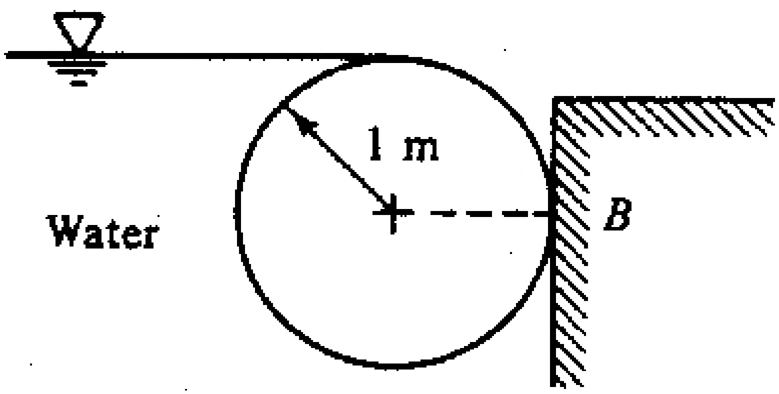
$$= (1.230 \times 10^5)\ (0.0866)$$

$$= 1.065 \times 10^5 \text{ N} \bullet \text{m}$$

$$= \underline{107 \text{ kM} \bullet \text{m}}$$

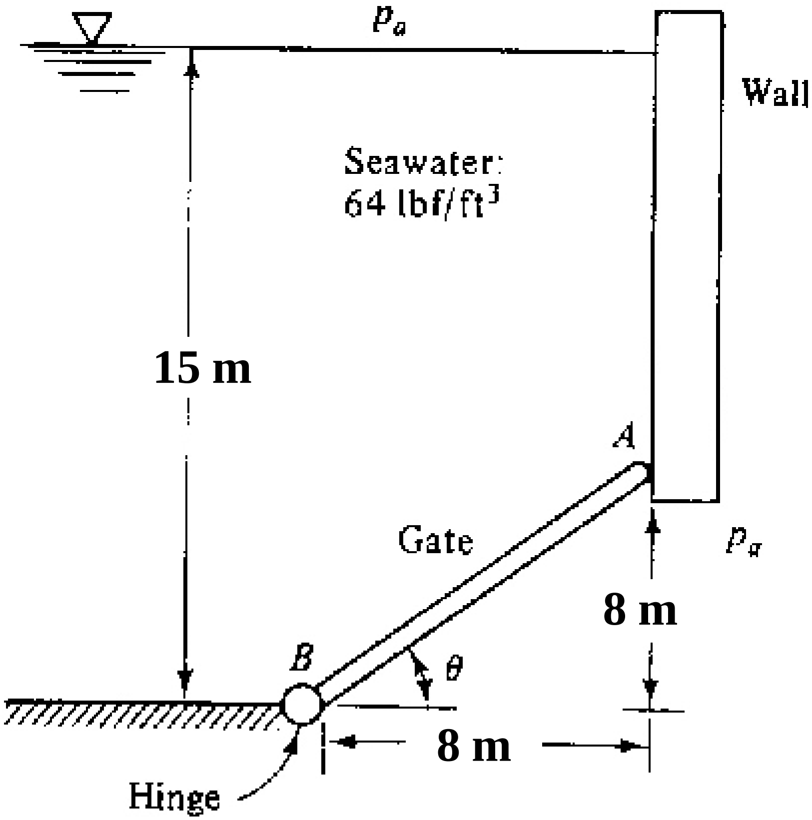
EXAMPLE

- The 2-m diameter cylinder shown is 5m long into the paper and rests in static equilibrium against smooth wall at point *B*.
- Compute the weight of the cylinder assume zero wall friction at point *B*



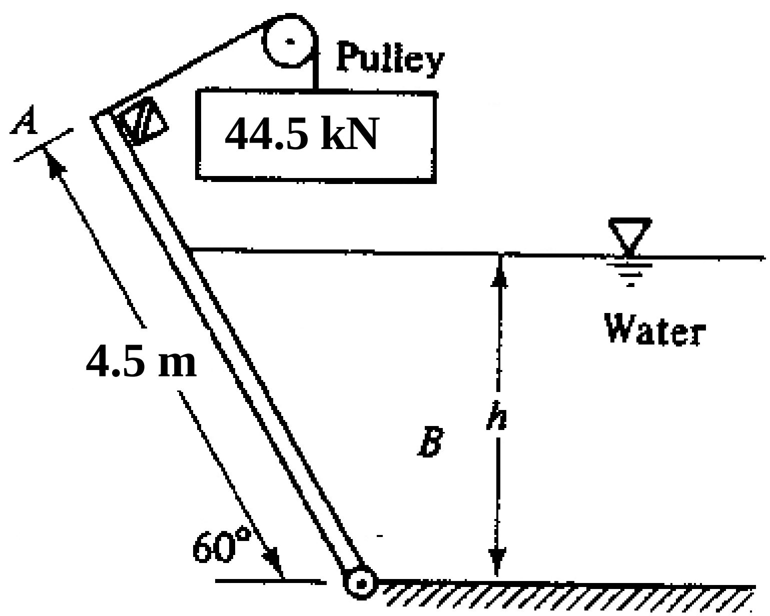
EXAMPLE

- A gate 5m wide is hinged at point *B* and rests against a smooth wall at point *A*, Compute:
- The force on the gate due to water
- The horizontal force P exerted by the wall at point *A*



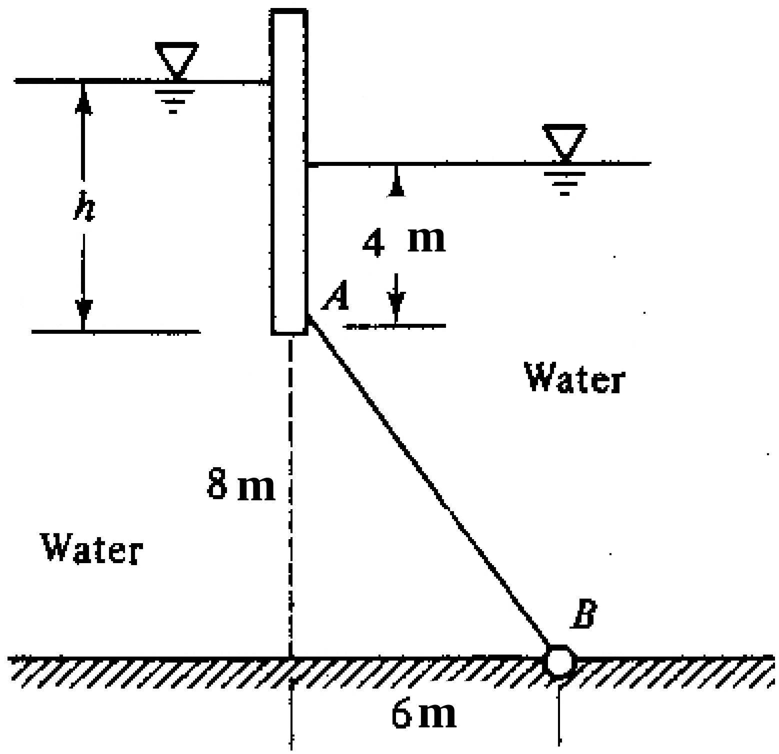
EXAMPLE

- The gate *AB* is 4.5 m long and 3 m wide into the paper and hinged at B with a stop at *A*. Neglecting the weight of the gate, compute the water level *h* for which the gate will start to fall



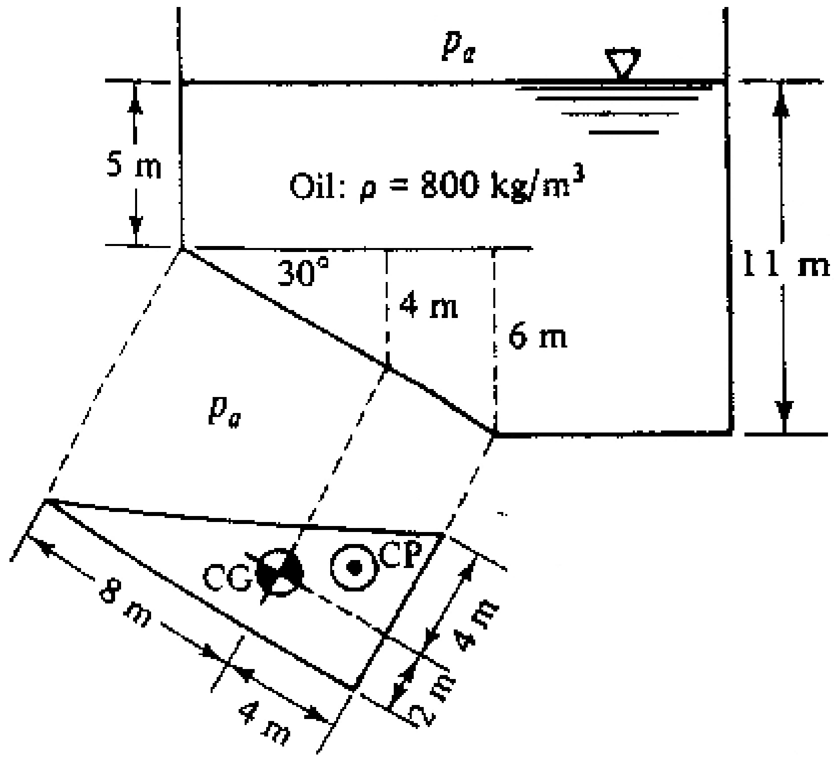
EXAMPLE

- Gate *AB* is 5 m into the paper and weights 30 kN when submerged.
- It is hinged at *B* and rests against a smooth wall at *A*.
- determine the water level ***h*** at the left which will just cause the gate to open

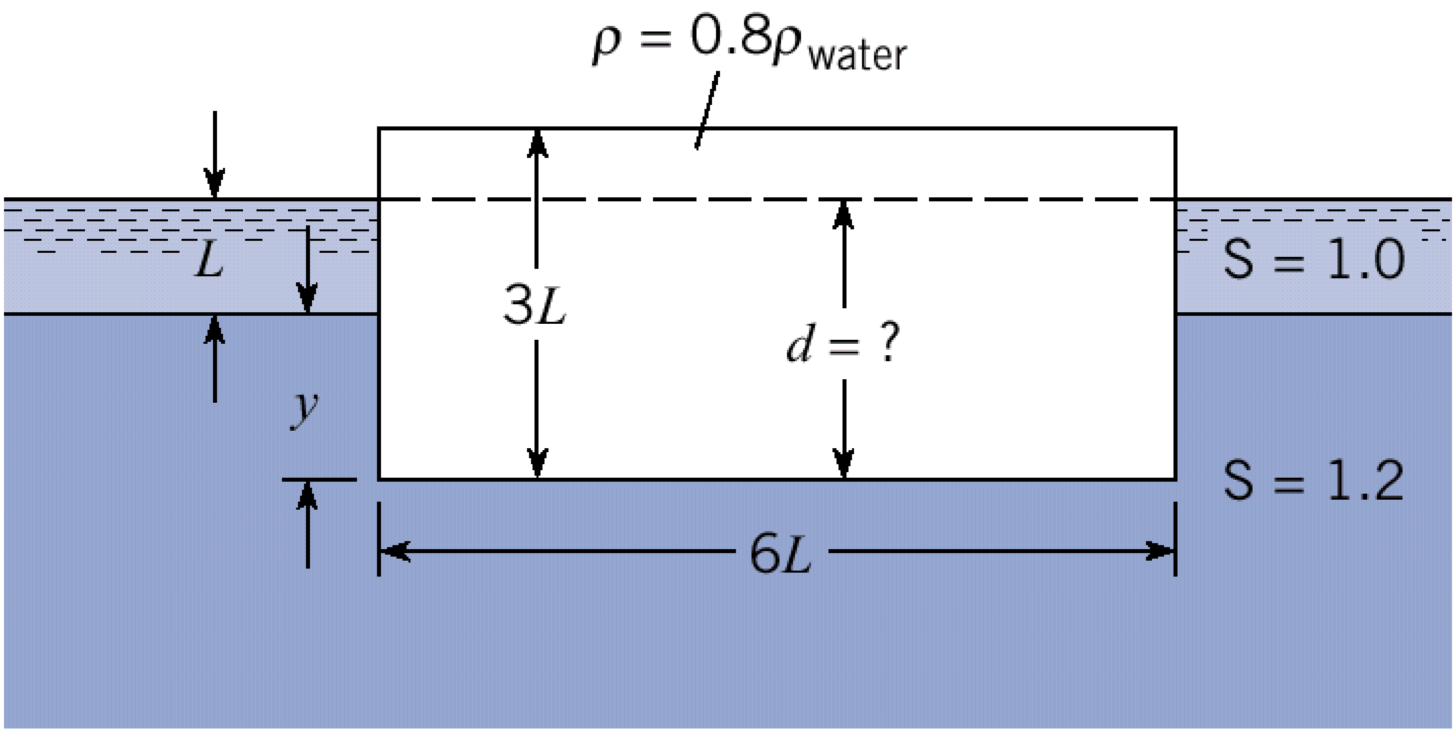


EXAMPLE

- A tank has right triangular gate near the bottom, Compute:
- The hydrostatic force on the gate
- the center of pressure on the gate

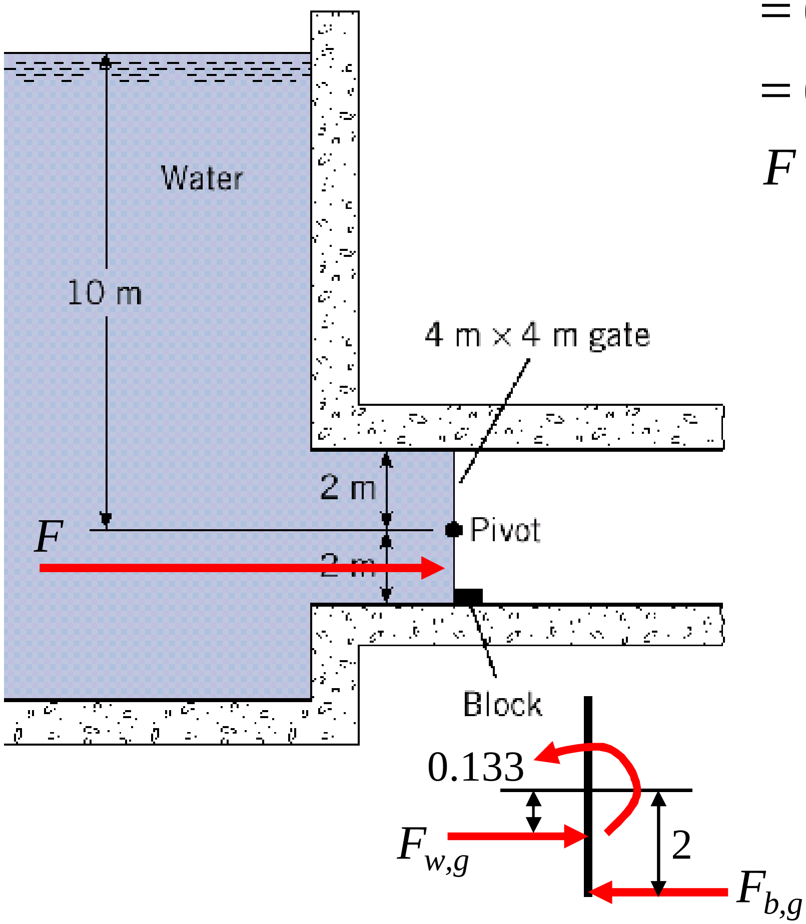


Example



Example

Find: Force of block on gate

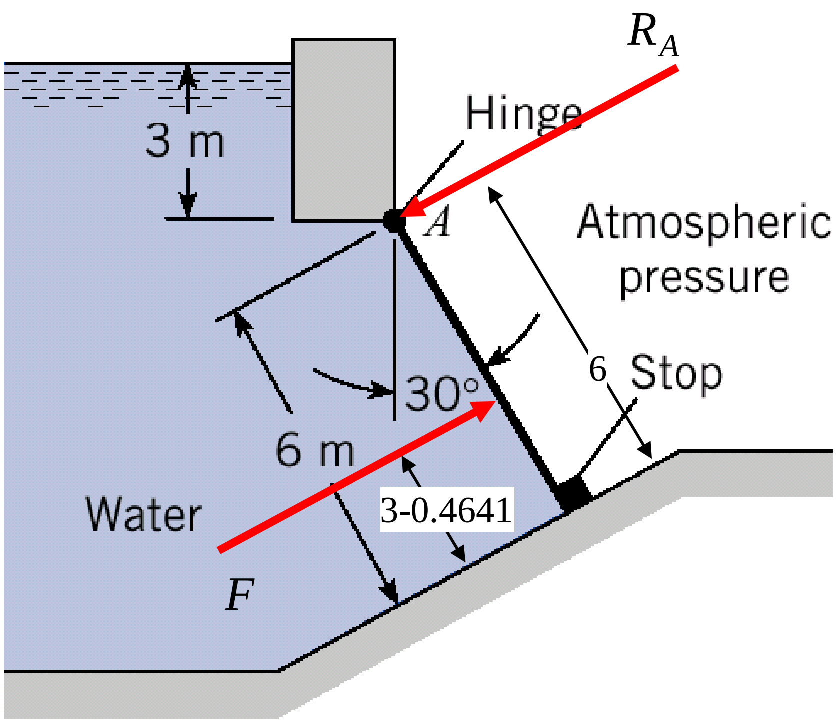


$$F = \bar{p}A$$
$$= (g\bar{y}\sin a)A$$
$$= (9810*10*1)*(4*4)$$
$$F = 1569.6\,kN$$

$$y_{cp} - \bar{y} = \frac{\bar{I}}{\bar{y}A}$$
$$= \frac{4*4^3/12}{(10*4*4)}$$
$$= 0.133\,m$$

$$\sum M = 0$$
$$= 0.133F_{w,g} - 2F_{b,g}$$
$$F_{b,g} = \frac{0.133}{2}F_{w,g}$$
$$= \frac{0.133}{2}1569.6\,kN$$
$$F_{b,g} = 104.378\,kN$$

Example



$$F = \bar{p}A = (g\bar{y}\sin a)A$$
$$= 9810 * (3 + 3\cos 30) * (4 * 6)$$
$$= 1{,}318{,}000\,N$$

$$y_{cp} - \bar{y} = \frac{\bar{I}}{\bar{y}A} = \frac{4 * 6^3 / 12}{(6.464 * 24)}$$
$$= 0.4641 m$$

$$\sum M = 0$$
$$= 6R_A - (3 - 0.4641)F$$
$$R_A = \frac{3 - 0.4641}{6} F$$
$$= (0.42265)1318\,kN$$
$$R_A = 557.05\,kN$$